3. The specified composition product for closed braids

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Since u is easily seen to define a bijection from

$$\{(f, g)/f \in L(D, 2), g \in L(D_{f,1}, q)\}$$

to $L(D, q+1)$ we obtain

$$H'(D, z, t^{q+1}) = t^{-(q+2)r(D)} \sum_{h \in L(D, q+1)} <D \mid h> t^{w(D,h)+2s(D,h)}$$

as required.

Remarks. (1) The case $q = 2$ in Proposition 2 yields a state model for the Jones polynomial which, as noted in [28], can be directly related to Kauffman's "bracket polynomial" model [17, 18] via the theory of "ice-type models" developed in [2], Section 12.3.

(2) The state models of Proposition 2 can be used as shown in [28] to obtain a proof of the existence of the homfly polynomial.

(3) The applicability of Proposition 1 is limited by the fact that it cannot deal efficiently with the Alexander-Conway polynomial. This is because for every non-empty diagram D, $H'(D, z, 1) = 0$. Another aspect of this phenomenon is that property (v) cannot be defined in a coherent way for $H(D, z, 1) = A(D, z)$: the effect of adding a free loop to a non-empty diagram is qualitatively different from the corresponding effect on the empty diagram. However, a coherent version of property (v) is essential to the proof of Proposition 1. This has lead us to look for another form of the composition product which will be capable of handling the Alexander-Conway polynomial. So far we have been able to define such a composition product only in the case of closed braids. This is presented in the next section.

3. The specified composition product for closed braids

3.1. Braid words, braid diagrams and the specified product

Let us consider an infinite sequence of symbols (s_i) indexed by the set of positive integers. Artin's braid group on n strings $B_n (n \geqslant 1)$ can be defined by the presentation:

$$<s_1, \dots s_{n-1} \mid s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, i = 1, \dots n-2; s_i s_j = s_j s_i, \mid i-j \mid \geqslant 2> .$$

Thus B_1 is the trivial group and B_n is the subgroup of B_{n+1} generated by $s_1, \dots s_{n-1}$. We call *braid word on n strings* any word on the alphabet $\{s_i, s_i^{-1}/i = 1, \dots n-1\}$. Thus a braid word on n strings is also a braid word on n' strings for all $n' \geqslant n$. To every braid word m on n strings we

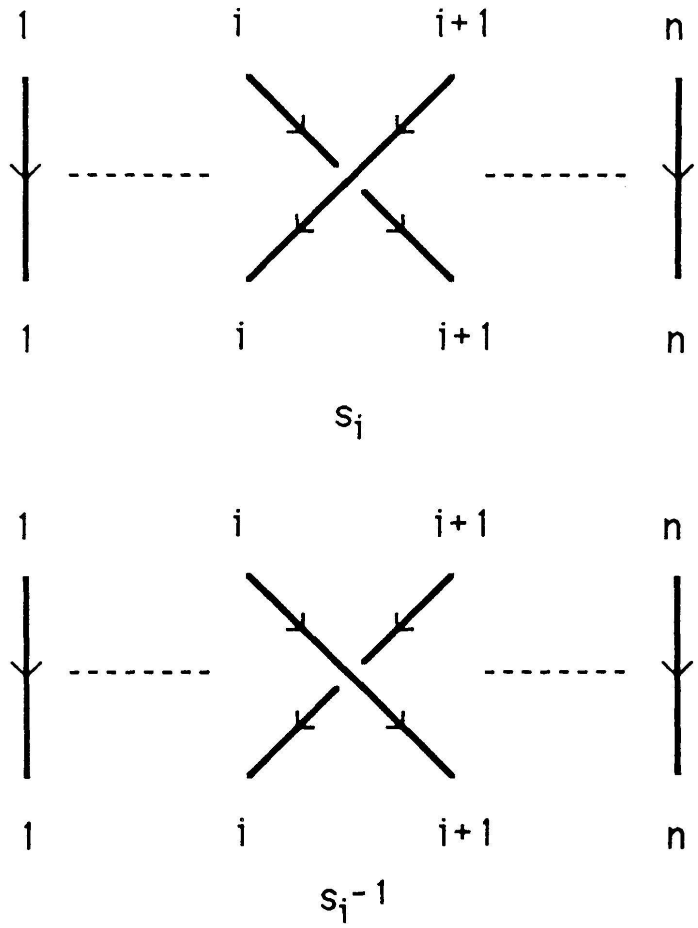

FIGURE 23

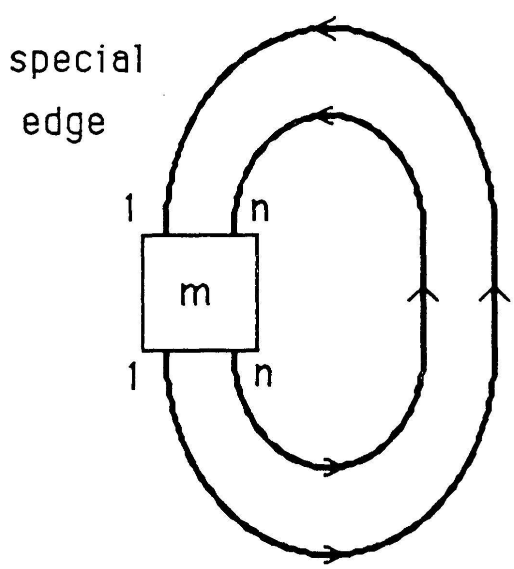

FIGURE 24

associate a *braid diagram on n strings* as follows. To each letter of m corresponds a portion of diagram, or *block*, according to the rule described on Figure 23. Each block has n top incoming *half-edges*, numbered from 1 to n in the left-to-right order, and n bottom outgoing half-edges which are numbered similarly. First the blocks are concatenated from top to bottom in the order of occurrence of the corresponding letters in m. Here the concatenation of two blocks corresponds to the merging of the bottom half-edges of the first block to the top half-edges with corresponding numbers of the second block. Finally the bottom half-edges are merged as shown on Figure 24 to the corresponding top half-edges, thus forming *n return edges.* The return edges will be numbered from 1 to n as the corresponding half-edges. The return edge which is numbered 1 will be called the *special edge.* We note that a braid diagram is non empty, and that the unique braid diagram on one string is a free loop. We also observe that any braid diagram on n strings has rotation number n.

A *specified labelling* of the braid diagram D is a labelling such that the special edge receives the label 1. We denote by $SL(D, k)$ the set of specified labellings of D which take their values in $\{1, \dots k\}$.

PROPOSITION 3. *For any braid diagram* D,

$$\sum_{f\in SL(D,2)} \langle D \mid f\rangle\, a_2^{-r(D_{f,1})+1}\, a_1^{r(D_{f,2})}\, H(D_{f,1}, z, a_1)\, H'(D_{f,2}, z, a_2) = H(D, z, a_1 a_2)\,.$$

Proof. Let $H''(D, z, a_1, a_2)$ denote the expression

$$\sum_{f\in SL(D,2)} \langle D \mid f\rangle\, a_2^{-r(D_{f,1})+1}\, a_1^{r(D_{f,2})}\, H(D_{f,1}, z, a_1)\, H'(D_{f,2}, z, a_2)$$

which is an element of the ring K of Laurent polynomials with integer coefficients in the variables z, a_1, a_2.

Let us introduce the following modifications in each step of the proof of Proposition 1: we replace $H'(D_{f,1}, z, a_1)$ by $H(D_{f,1}, z, a_1)$, we restrict our attention to specified labellings, and we assume that the special edge is outer. Then it is easy to check that, since $D_{f,1}$ is non-empty for every specified labelling f, our arguments (and especially those involving property (v) for $H(D_{f,1}, z, a_1)$) remain correct.

Thus in particular we have the following properties for braid diagrams on n strings D, D', D^+, D^-, D°:

(1) If D' is obtained from D by a move of type B_1, B_2 or C for which the return edges are outer (such a move will be called *inner*), then $H''(D', z, a_1, a_2) = H''(D, z, a_1, a_2)$.

(2) If (D^+, D^-, D°) form a Conway triple then:

$$H''(D^+, z, a_1, a_2) - H''(D^-, z, a_1, a_2) = zH''(D^\circ, z, a_1, a_2).$$

Clearly two braid words on n strings represent the same element of the group B_n if and only if the associated braid diagrams on n strings can be obtained one from the other by a finite sequence of inner moves of types B_1, B'_1, B_2, B'_2 or C, C' (the insertion or deletion of trivial relators $s_i s_i^{-1}$ or $s_i^{-1} s_i$ corresponds to the moves of types B_1, B'_1, B_2, B'_2; we associate in a similar way the relations $s_i s_j = s_j s_i$, $|i - j| \geqslant 2$ to plane deformations and the relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ to the moves of types C, C').

Hence it follows from (1) that we may define a mapping H''_n from B_n to K as follows. For every element b of B_n, $H''_n(b) = H''(D, z, a_1, a_2)$ for any braid diagram on n strings D associated to a braid word which represents b.

Consider now the quotient A_n of the group algebra $K(B_n)$ by the (two-sided) ideal J generated by the elements $s_i - s_i^{-1} - z\,(i=1, \dots, n-1)$. Let us extend H''_n to $K(B_n)$ by linearity. Then it follows from (2) that H''_n takes the same value on two elements of $K(B_n)$ which are congruent modulo J. We shall now identify H''_n with the induced linear functional from A_n to K. The embedding of B_n into B_{n+1} is extended to an embedding of A_n into A_{n+1} in the obvious way. Thus H''_n is well defined on $A_1, \dots A_n$.

Coming back to our modified version of the proof of Proposition 1, we obtain the following properties for all $n \geqslant 1$.

(3) If $u \in A_n$ then $H''_{n+1}(u) = dH''_n(u)$ with $d = \left(a_1 a_2 - (a_1 a_2)^{-1}\right) z^{-1}$.

(4) If $u, v \in A_n$ then $H''_{n+1}(us_n v) = aH''_n(uv)$ and $H''_{n+1}(us_n^{-1} v) = a^{-1} H''_n(uv)$ with $a = a_1 a_2$.

Indeed it is enough to prove (3) and (4) for u, v in B_n. Then (3) corresponds to the addition of a single free loop to a non-empty diagram, and (4) to a move of type A'_1 or A'_2.

It is also easily checked that when D is the unique braid diagram on one string, that is, the free loop with rotation number 1, $H''(D, z, a_1, a_2) = 1$. Since B_1 is trivial, A_1 can be identified with K. Hence

(5) H''_1 is the identity mapping on K.

We now show that

(6) If $u, v \in A_n$ then $H''_n(uv) = H''_n(vu)$.

The following proof is essentially the same as those given in [7] and [13] for Ocneanu's trace [25]. However since the context is slightly different we feel necessary to give the details. We proceed by induction on n. The result is trivial for $n = 1$ by (5). Assume that the result holds for A_n. It is enough to show that for all u in A_{n+1} and i in $1, \ldots, n$, $H''_{n+1}(us_i) = H''_{n+1}(s_i u)$. We need the following basic result: A_{n+1} is generated as a K-algebra by $G_{n+1} = A_n \cup \{xs_n y/x, y \in A_n\}$ (see [7], [13], [31]). Now we may assume that u belongs to G_{n+1}.

If $u \in A_n$ and $i < n$, by our induction hypothesis $H''_n(us_i) = H''_n(s_i u)$ and property (3) yields the result. Similarly by property (4), for u in A_n, $H''_{n+1}(us_n) = aH''_n(u) = H''_{n+1}(s_n u)$.

Assume now that $u = xs_n y$ with $x, y \in A_n$. If $i < n$, $H''_{n+1}(xs_n ys_i) = aH''_n(xys_i)$ and $H''_{n+1}(s_i xs_n y) = aH''_n(s_i xy)$; the result then follows from the induction hypothesis. It remains to prove that $H''_{n+1}(xs_n ys_n) = H''_{n+1}(s_n xs_n y)$. We may assume that $x, y \in G_n$. There are four cases to consider.

— If x and y both belong to A_{n-1} they commute with s_n and the result is immediate.

— If $x = x's_{n-1}x''$ with $x', x'' \in A_{n-1}$:

$$s_n xs_n y = s_n x's_{n-1}x''s_n y = x's_n s_{n-1}s_n x''y = x's_{n-1}s_n s_{n-1}x''y .$$

Then by property (4)

$$\begin{aligned} H''_{n+1}(s_n xs_n y) &= aH''_n(x's^2_{n-1}x''y) = aH''_n(x'(zs_{n-1}+1)x''y) \\ &= azH''_n(xy) + aH''_n(x'x''y) . \end{aligned}$$

Now if $y \in A_{n-1}$

$$xs_n ys_n = xys^2_n = xy(zs_n + 1) .$$

Then by properties (3) and (4)

$$H''_{n+1}(xs_n ys_n) = zaH''_n(xy) + dH''_n(xy) .$$

The result follows since $dH''_n(x's_{n-1}x''y) = aH''_n(x'x''y) = adH''_{n-1}(x'x''y)$.

On the other hand if $y = y's_{n-1}y''$ with $y', y'' \in A_{n-1}$

$$xs_n ys_n = xs_n y's_{n-1}y''s_n = xy's_n s_{n-1}s_n y'' = xy's_{n-1}s_n s_{n-1}y''$$

and hence

$$\begin{aligned} H''_{n+1}(xs_n ys_n) &= aH''_n(xy's^2_{n-1}y'') = aH''_n(xy'(zs_{n-1}+1)y'') \\ &= azH''_n(xy) + aH''_n(xy'y'') . \end{aligned}$$

The result follows since

$$H''_n(x'x''y) = H''_n(x'x''y's_{n-1}y'') = aH''_{n-1}(x'x''y'y'') = H''_n(x's_{n-1}x''y'y'') = H''_n(xy'y'') .$$

— In the only remaining case $x \in A_{n-1}$ and $y = y's_{n-1}y''$ with $y', y'' \in A_{n-1}$. We have just seen that $H''_{n+1}(xs_nys_n) = azH''_n(xy) + aH''_n(xy'y'')$. We may also write

$$s_nxs_ny = s_n^2xy = (zs_n+1)xy ,$$

so that $H''_{n+1}(s_nxs_ny) = zaH''_n(xy) + dH''_n(xy)$.

The result follows since $dH''_n(xy's_{n-1}y'') = aH''_n(xy'y'') = adH''_{n-1}(xy'y'')$.

Now we can use the construction of the homfly polynomial described in [7], [13]. Let us consider two braid diagrams D_1 on n_1 strings and D_2 on n_2 strings. For $i = 1, 2$ let b_i be the element of the braid group on n_i strings associated to D_i. Markov's theorem asserts that D_1 and D_2 are isotopic if and only if b_1 can be obtained from b_2 by a finite sequence of moves of one of the following types:

Markov move of type 1: replace $b \in B_n$ by a conjugate $cbc^{-1} (c \in B_n)$.

Markov move of type 2: replace $b \in B_n$ by $bs_n \in B_{n+1}$ or $bs_n^{-1} \in B_{n+1}$, or perform the converse operation.

It then follows from properties (4) and (6) that if D_1 and D_2 are isotopic:

$$(a_1a_2)^{-w(D_1)} H''(D_1, z, a_1, a_2) = (a_1a_2)^{-w(D_2)} H''(D_2, z, a_1, a_2) .$$

Since by the classical result of Alexander every diagram is isotopic to some braid diagram, there exists an isotopy invariant $P''(D, z, a_1, a_2)$ defined for all diagrams D which is equal to $(a_1a_2)^{-w(D)} H''(D, z, a_1, a_2)$ for every braid diagram D. Using property (2) and a refinement of Alexander's Theorem one easily shows as in [7] p. 294 or [13] p. 348 that this invariant satisfies the equation

$$a_1a_2 P''(D^+, z, a_1, a_2) - (a_1a_2)^{-1} P''(D^-, z, a_1, a_2) = zP''(D^\circ, z, a_1, a_2)$$

for every Conway triple (D^+, D^-, D°). Then property (5) allows us to identify $P''(D, z, a_1, a_2)$ with the homfly polynomial $P(D, z, a_1a_2)$, and thus to complete the proof.

Remark. In Proposition 3 we may replace $r(D_{f,i})$ by the number of return edges labelled i by f.

3.2. A SIMPLIFIED STATE MODEL

We give here a "specified" version of Proposition 2. As before we write $z = t - t^{-1}$.

PROPOSITION 4. *For any braid diagram D and positive integer q,*

$$H(D, z, t^q) = t^{q-1-(q+1)r(D)} \sum\nolimits_{f \in SL(D, q)} \langle D \mid f \rangle \, t^{w(D, f) + 2s(D, f)},$$

where $w(D, f) = \sum_{i=1, \dots q} w(D_{f, i})$ *and* $s(D, f) = \sum_{i=1, \dots q} ir(D_{f, i})$.

The proof is almost identical to that of Proposition 2 and will be omitted. Note that $H(D, z, t^q)$ is now expressed as a Laurent polynomial in t (this was not the case in Proposition 2 which only gave such an expression for $H'(D, z, t^q)$).

3.3. MODELS FOR THE ALEXANDER-CONWAY POLYNOMIAL

We begin with the following identity which is immediately obtained by setting $a_1 = a$ and $a_2 = a^{-1}$ in Proposition 3:

PROPOSITION 5. *For any braid diagram D,*

$$A(D, z) = a^{r(D)-1} \sum\nolimits_{f \in SL(D, 2)} \langle D \mid f \rangle \, H(D_{f, 1}, z, a) \, H'(D_{f, 2}, z, a^{-1}).$$

Remark: For a diagram D, let $D^{\sim}$ denote the *mirror image* of D, that is, the diagram obtained from D by changing the signs of all vertices. It is easy to deduce from the Theorem of section 2.1 that $H(D^{\sim}, z, a) = H(D, -z, a^{-1}) = (-1)^{c(D)-1} H(D, z, a^{-1})$, where $c(D)$ denotes the number of components of D (in the sense of knot theory, not graph theory). These identities can be used to reformulate Proposition 5.

We now recall that for $z = t - t^{-1}, H(D, z, t) = t^{w(D)}$. Similarly, $H(D, z, -t^{-1}) = (-t)^{-w(D)}$. It easily follows that

$$H'(D, z, t^{-1}) = (-1)^{r(D)} (-t)^{-w(D)}.$$

Taking $a = t$ in Proposition 5 we obtain:

PROPOSITION 6. *For any braid diagram D,*

$$A(D, z) = t^{r(D)-1} \sum\nolimits_{f \in SL(D, 2)} \langle D \mid f \rangle \, t^{w(D_{f, 1})} (-t)^{-w(D_{f, 2})} (-1)^{r(D_{f, 2})}.$$

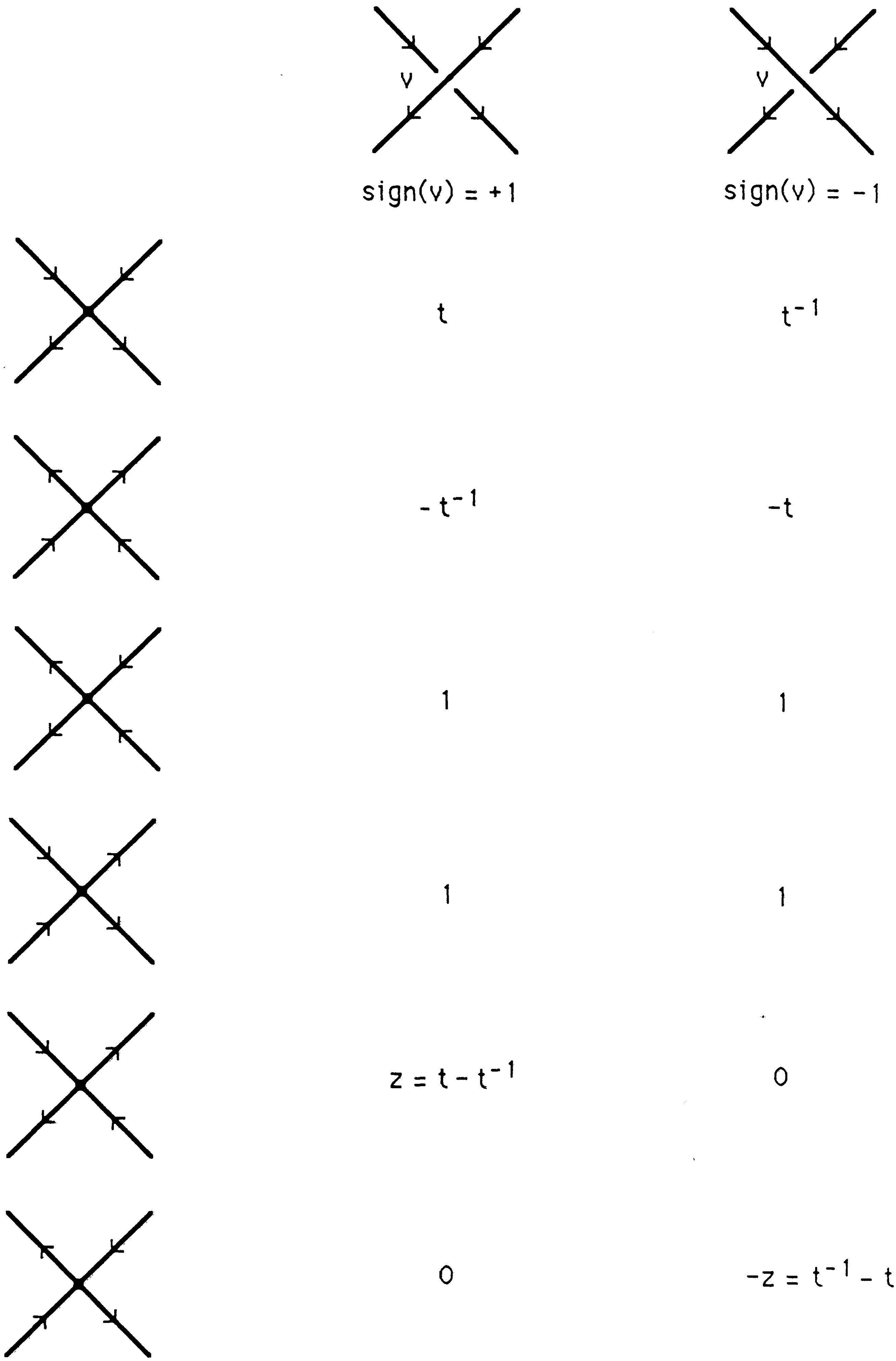

FIGURE 25

Remark. In the above expression for $A(D, z)$ we may replace t by $-t^{-1}$.

Proposition 6 yields an "ice-type" model (see [2]) for the Alexander-Conway polynomial. Let us call *Eulerian orientation* of a diagram D every rearrangement of the edge orientations such that at every vertex v the number of edges incident towards v equals the number of such edges incident from v (a loop at v contributing 1 to both numbers). We shall denote by $O(D)$ the set of Eulerian orientations of D. It is easy to see that for every labelling f in $L(D, 2)$ if we reverse the orientations of all edges labelled 2 we obtain an Eulerian orientation of D, and that this defines a bijective correspondence from $L(D, 2)$ to $O(D)$. Moreover when D is a braid diagram, if we denote by $SO(D)$ the set of Eulerian orientations of D such that the orientation of the special edge is not changed, we obtain a bijection from $SL(D, 2)$ to $SO(D)$. For a vertex v of D and an Eulerian orientation o of D let us define their *interaction* $<v \mid o>$ as on Figure 25. If D has vertex-set V let us write $<D \mid o> = \prod_{v \in V} <v \mid o>$. Finally let $r(o)$ denote the number of return edges of D which are reversed in the orientation o. Then we may reformulate Proposition 6 as follows.

PROPOSITION 7. *For any braid diagram* D *on* n *strings,*

$$A(D, z) = t^{n-1} \sum_{o \in SO(D)} (-1)^{r(o)} <D \mid o>$$

3.4. THE ALEXANDER EXPANSION FOR THE HOMFLY POLYNOMIAL

We begin with the following immediate consequence of Proposition 3.

PROPOSITION 8. *For any braid diagram* D,

$$H(D, z, a) = \sum_{f \in SL(D, 2)} <D \mid f> a^{-r(D_{f,1})+1} A(D_{f,1}, z) H'(D_{f,2}, z, a) .$$

Let D be a braid diagram on n strings and denote its return edges by $e_1, \dots e_n$ in left to right order, e_1 being the special edge. For a labelling f of D, let us call *pattern* of f the partition of $\{e_1, \dots e_n\}$ defined by all the non-empty $f^{-1}(i)$. We shall call a labelling f of D *compressed* if the word $f(e_1) \dots f(e_n)$ is lexicographically minimal (for the usual ordering of the integers) in the class of all words of the form $f'(e_1) \dots f'(e_n)$ where f' is a labelling with the same pattern as f. We denote by $CL(D)$ the set of compressed labellings of D and by $k(f)$ the cardinality of the image of the labelling f.

PROPOSITION 9. *For any braid diagram* D,

$$H(D, z, a) = \sum_{f \in CL(D)} <D \mid f> a^{-r(D)+k(f)} \left((a-a^{-1})z^{-1}\right)^{k(f)-1} \prod_{i=1,\dots k(f)} A(D_{f,i}, z).$$

Proof. We proceed by induction on the rotation number of D. The result is trivial if this number is 1. Now by Proposition 8, $H(D, z, a) = \sum_{f \in SL(D,2)} C(D, f)$, with

$$C(D, f) = <D \mid f> a^{-r(D_{f,1})+1} A(D_{f,1}, z) H'(D_{f,2}, z, a).$$

Let us fix f and write $D2$ for $D_{f,2}$. If $D2$ is empty, $H'(D2, z, a) = 1$. Otherwise $D2$ is a braid diagram whose special edge is the image under the projection P_f associated to f of the leftmost return edge of D which is labelled 2 by f. Since $D_{f,1}$ is not empty, $r(D2) < r(D)$. Then by our induction hypothesis, $H'(D2, z, a) = (a-a^{-1})z^{-1}H(D2, z, a)$ is equal to $\sum_{g \in CL(D2)} <D2 \mid g> a^{-r(D2)+k(g)} \left((a-a^{-1})z^{-1}\right)^{k(g)} \prod_{i=1,\dots k(g)} A(D2_{g,i}, z)$. Thus if $D2$ is not empty, since $r(D_{f,1}) + r(D2) = r(D)$, $C(D, f)$ is equal to

$$\sum_{g \in CL(D2)} <D \mid f> <D2 \mid g> a^{-r(D)+k(g)+1} \left((a-a^{-1})z^{-1}\right)^{k(g)} A(D_{f,1}, z) \prod_{i=1,\dots k(g)} A(D2_{g,i}, z).$$

For every g in $CL(D2)$, define a labelling $h = u(f, g)$ of D as follows. For an edge e of D, if $f(e) = 1$ then $h(e) = 1$; if $f(e) = 2$ then $h(e) = g(P_f(e)) + 1$, where P_f is the projection associated to f. Since f is specified and g is compressed, h is easily seen to be compressed. Clearly $k(h) = k(g) + 1$, $D_{f,1} = D_{h,1}$ and for $i = 1, \dots k(g)$, $D2_{g,i} = D_{h,i+1}$. Moreover h is compatible with f, the constant labelling of $D1$ with value 1 and the labelling g of $D2$. Hence, by the Unification Lemma, $<D \mid f> <D2 \mid g> = <D \mid h>$.

It follows that, if $D2$ is not empty, $C(D, f)$ is equal to

$$\sum_{g \in CL(D2),\, h=u(f,g)} <D \mid h> a^{-r(D)+k(h)} \left((a-a^{-1})z^{-1}\right)^{k(h)-1} \prod_{i=1,\dots k(h)} A(D_{h,i}, z).$$

For any compressed labelling h of D define

$$C'(D, h) = <D \mid h> a^{-r(D)+k(h)} \left((a-a^{-1})z^{-1}\right)^{k(h)-1} \prod_{i=1,\dots k(h)} A(D_{h,i}, z).$$

Denoting by f_0 the labelling of D which takes the value 1 on each edge, it remains to prove:

$$\sum_{h \in CL(D)} C'(D, h) = C(D, f_0) + \sum_{f \in SL(D,2),\, k(f)=2} \sum_{g \in CL(D2),\, h=u(f,g)} C'(D, h).$$

It is easy to check that $C'(D, f_0) = C(D, f_0) = a^{-r(D)+1} A(D, z)$. Moreover u

defines a bijection from $\{(f, g)/f \in SL(D, 2), k(f) = 2, g \in CL(D_{f,2})\}$ to $CL(D) - \{f_0\}$. This completes the proof.

3.5. Some consequences

Let D be a braid diagram on n strings and consider its homfly polynomial $P(D, z, a) = a^{-w(D)} H(D, z, a)$. Let $E(D)$ (respectively: $e(D)$) be the maximum (respectively: minimum) degree in the variable a of the Laurent polynomial $P(D, z, a)$. The following result is due to Franks and Williams [4] and Morton [22]. In fact in [22] it is generalized to arbitrary diagrams (the number of Seifert circles replacing the number of strings).

Proposition 10. *For any braid diagram D on n strings,*

$$1 - n - w(D) \leqslant e(D) \leqslant E(D) \leqslant n - 1 - w(D).$$

Proof. Using Proposition 9 we may write

$$P(D, z, a) = \sum\nolimits_{f \in CL(D)} Q(D, f, z)\, a^{-w(D)-n+1} (a^2 - 1)^{k(f)-1},$$

where $Q(D, f, z) = \langle D \mid f \rangle z^{-k(f)+1} \prod_{i=1,\dots k(f)} A(D_{f,i}, z)$ is a Laurent polynomial in z. The result follows immediately since $1 \leqslant k(f) \leqslant n$.

Remark. For fixed z and n the fact that $H(D, z, a)$ is up to a constant factor a polynomial of degree at most $n - 1$ in the variable $a^2 - 1$ can be used to express it as a linear combination of the form $\sum_{i=1,\dots n} \lambda_i H(D, z, a_i)$ for n mutually independent variables a_i. The coefficients λ_i are given explicitly by Murakami in [23].

The following result is the specialization to braid diagrams of another result of Morton [22]. Let us denote by $M(D)$ the maximum degree in the variable z of the Laurent polynomial $P(D, z, a)$ (or equivalently $H(D, z, a)$).

Proposition 11. *For any braid diagram D on n strings with vertex-set V,*

$$M(D) \leqslant |V| - n + 1.$$

Proof. Coming back to Proposition 6, we see that every term in the expansion of $A(D, z)$ in powers of t is, up to sign, of the form

$$t^{n-1}(t - t^{-1})^{h(D,f)}\, t^{w(D_{f,1}) - w(D_{f,2})}$$

for some f in $SL(D, 2)$, where $h(D, f)$ is the number of vertices where $D_{f,1}$ and $D_{f,2}$ are mutually tangent. The minimum degree of such a term is not less than $n - 1 - |V|$. Hence the maximum degree in z of $A(D, z)$ is not greater than $|V| - n + 1$ (this result is generalized to arbitrary diagrams in [16] p. 120).

Now applying this to all the $D_{f,i}$ in the expression

$$Q(D, f, z) = \langle D \mid f \rangle \, z^{-k(f)+1} \prod_{i=1,\dots k(f)} A(D_{f,i}, z)$$

and noting that the vertices of the $D_{f,i}$ contribute 1 to the product $\langle D \mid f \rangle$ we easily obtain the desired inequality.

3.6. A STATE MODEL FOR THE HOMFLY POLYNOMIAL

We may combine Propositions 6 and 9 to obtain a state model for the homfly polynomial.

Consider a compressed labelling f of the braid diagram D together with a specified labelling g_i in $SL(D_{f,i}, 2)$ for each $i = 1, \dots k(f)$, and define the labelling $h = u(f, g_1, \dots g_{k(f)})$ as follows. For every edge e of D, if $f(e) = i$ then $h(e) = 2(i-1) + g_i(P_f(e))$, where P_f is the projection associated to f. Any labelling which can be obtained in this way will be called *half-compressed* and we shall denote by $C'L(D)$ the set of half-compressed labellings of D. For a labelling f of D we denote by $w_o(D, f)$ (respectively: $w_e(D, f)$) the sum $\sum_{i=1,\dots k(f)} w(D_{f,i})$ restricted to the odd (respectively: even) values of i. Similarly we define $r_e(D, f)$ as the sum $\sum_{i=1,\dots k(f)} r(D_{f,i})$ restricted to the even values of i. We denote by $k'(f)$ the number of distinct odd values taken by f.

PROPOSITION 12. *For any braid diagram* $D, H(D, z, a)$ *equals*

$$\sum_{f \in C'L(D)} \langle D \mid f \rangle \, (t^{-1}a)^{-r(D)+k'(f)} \left((a-a^{-1})z^{-1}\right)^{k'(f)-1} t^{w_o(D,f)}$$
$$(-t)^{-w_e(D,f)} (-1)^{r_e(D,f)}.$$

Proof. Let f be a fixed compressed labelling of D. For each $i = 1, \dots k(f)$ let us write Di for $D_{f,i}$. By Proposition 6,

$$A(Di, z) = t^{r(Di)-1} \sum_{gi \in SL(Di, 2)} \langle Di \mid gi \rangle \, t^{w(Di_{gi,1})} (-t)^{-w(Di_{gi,2})} (-1)^{r(Di_{gi,2})}.$$

Let $T(D, f)$ be the set of $k(f)$-tuples $g = (gi, i=1, \dots k(f))$ with $gi \in SL(Di, 2)$ for all i. Then $\langle D \mid f \rangle \prod_{i=1,\dots k(f)} A(Di, z) =$

$$t^{r(D)-k(f)} \sum_{g \in T(D,f)} \langle D \mid f \rangle \prod_{i=1,\dots k(f)} \langle Di \mid gi \rangle \, t^{w(Di_{gi,1})}$$
$$(-t)^{-w(Di_{gi,2})} (-1)^{r(Di_{gi,2})}$$

Clearly for $h = u(f, gi(i=1, \ldots k(f))) = u(f, g)$:

$$\sum_{i=1,\ldots k(f)} w(Di_{gi,1}) = w_o(D, h), \sum_{i=1,\ldots k(f)} w(Di_{gi,2}) = w_e(D, h)$$

and

$$\sum_{i=1,\ldots k(f)} r(Di_{gi,2}) = r_e(D, h).$$

Moreover, since h is easily seen to be compatible with f and the gi, it follows from the Unification Lemma that $<D \mid f> \prod_{i=1,\ldots k(f)} <Di \mid gi> = <D \mid h>$. Hence

$$<D \mid f> \prod_{i=1,\ldots k(f)} A(Di, z) =$$
$$t^{r(D)-k(f)} \sum_{g\in T(D,f),\, h=u(f,g)} <D \mid h>\, t^{w_o(D,h)} (-t)^{-w_e(D,h)} (-1)^{r_e(D,h)}$$

It now follows from Proposition 9 that

$$H(D, z, a) = \sum_{f\in CL(D)} \sum_{g\in T(D,f),\, h=u(f,g)}$$
$$<D \mid h>\, (t^{-1}a)^{-r(D)+k(f)} \left((a-a^{-1})z^{-1}\right)^{k(f)-1} t^{w_o(D,h)} (-t)^{-w_e(D,h)} (-1)^{r_e(D,h)}$$

In this expression $k(f)$ can be replaced by $k'(h)$. Moreover u clearly defines a bijection from the set $\{(f, g)/f \in CL(D),\ g \in T(D, f)\}$ to $C'L(D)$. This completes the proof.

4. Concluding remarks

1. It would be interesting to generalize the results of Section 3 to arbitrary diagrams. This is done for Proposition 6 in a joint paper with Louis Kauffman (in preparation).

(2) Since the topological and algebraic aspects of the Alexander polynomial are well understood, one may try to use Proposition 9 to gain some insight of the same kind on the homfly polynomial. Clearly one can combine Proposition 9 with classical results which relate the polynomial $A(D, z)$ to Burau matrices, Seifert surfaces and matrices, presentations of the fundamental group of the complement... This leads to corresponding complex labelled structures which seem to be worth studying. As for the combinatorial aspects Proposition 12 is only a first step and some further progress closely connected with Proposition 9 is reported in the following forthcoming papers: [A] Circuit partitions and the homfly polynomial of closed braids, Trans. AMS, to appear, [B] A combinatorial model for the homfly polynomial, preprint.