Section 4. Chern classes for locally finite groups

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is commutative and as both δ's are isomorphisms it suffices to show that

$$\delta^{-1} \circ c_1 : \operatorname{Hom}(G, \mathbf{C}^*) \to H^1(G, \mathbf{Q}/\mathbf{Z}) \simeq \operatorname{Hom}(G, \mathbf{Q}/\mathbf{Z})$$

is an isomorphism.

But by inspection $\delta^{-1} \circ c_1(\varphi) = \delta^{-1}(u) \circ \varphi$, and as u is a $\hat{\mathbf{Z}} = \operatorname{End}_{\mathbf{Z}}(\mu_\infty)$ generator for $H^2(\mu_\infty, \mathbf{Z}) \simeq \hat{\mathbf{Z}}$, $\delta^{-1}(u)$ is an isomorphism

$$\delta^{-1}(u) : \mu_\infty \xrightarrow{\sim} \mathbf{Q}/\mathbf{Z}.$$

Section 4. Chern classes for locally finite groups

The definition will be based on the following two observations. In the following, let $G = \varinjlim G_k$ be a locally finite group where $\{G_k\}$ is a family of finite subgroups.

Lemma. *Let*

$$\varphi : G \to Gl_n(\mathbf{C})$$

be a representation of G. *Then* φ *is uniquely determined by its restrictions*

$$\varphi_k : G_k \to Gl_n(\mathbf{C}).$$

Conversely given a family of compatible representations $\varphi_k : G_k \to Gl_n(\mathbf{C})$, *there exists a unique* $\varphi : G \to Gl_n(\mathbf{C})$ *which restricts to* φ_k *for all* k.

Proof. From the universal property of the direct limit, we have

$$\operatorname{Hom}(G, Gl_n(\mathbf{C})) \cong \varprojlim \operatorname{Hom}(G_k, Gl_n(\mathbf{C})).$$

Proposition. *For all* $i \geqslant 0$, *the natural map*

$$H^i(G, \mathbf{Z}) \cong \varprojlim H^i(G_k, \mathbf{Z}).$$

is an isomorphism.

Proof. Obvious for $i = 0, 1$. For $i \geqslant 1$, the homology groups

$$H_i(G, \mathbf{Z}) = \varinjlim H_i(G_k, \mathbf{Z})$$

are all abelian torsion groups.

Now by the universal coefficient theorem ($i \geqslant 1$)

$$H^i(G, \mathbf{Z}) \cong \mathrm{Ext}^1_{\mathbf{Z}}(H_{i-1}(G, \mathbf{Z}), \mathbf{Z})$$

and as observed above, for $i \geqslant 2$, $H_{i-1}(G, \mathbf{Z})$ is torsion. Thus

$$\begin{aligned} \mathrm{Ext}^1_{\mathbf{Z}}(H_{i-1}(G, \mathbf{Z}), \mathbf{Z}) &\cong \mathrm{Hom}_{\mathbf{Z}}(H_{i-1}(G, \mathbf{Z}), \mathbf{Q}/\mathbf{Z}) \\ \cong \mathrm{Hom}_{\mathbf{Z}}(\varinjlim H_{i-1}(G_k, \mathbf{Z}), \mathbf{Q}/\mathbf{Z}) &\cong \varprojlim \mathrm{Hom}_{\mathbf{Z}}(H_{i-1}(G_k, \mathbf{Z}), \mathbf{Q}/\mathbf{Z}) \\ &\simeq \varprojlim \mathrm{Ext}^1_{\mathbf{Z}}(H_{i-1}(G_k, \mathbf{Z}), \mathbf{Z}) \cong \varprojlim H^i(G_k, \mathbf{Z})\,. \end{aligned}$$

Combining these two results, there exists for a representation

$$\varphi : G \to Gl_n(\mathbf{C})$$

of a locally finite group $G = \varinjlim G_k$ a unique cohomology class $c.(\varphi) \in H^{**}(G, \mathbf{Z})$ such that for all k

$$\mathrm{res}^G_{G_k}(c.(\varphi)) = c.(\varphi_k)$$

Using this uniqueness result, it is easy to see that these classes satisfy the properties CH1, CH2 and CH3 and that they are uniquely determined by these properties.

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