8. Cauchy's residue theorem

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In order to calculate $\mathrm{Ind}(\gamma; s)$ we replace U by a small pointed neighbourhood D^* of s. With the notation of (7.2) let us write $\rho = \rho(s)\rho_s$ and deduce that

$$<\partial\gamma, \omega> \; = \; \rho(s)\mathrm{Tr}(\omega; s), \quad \omega \in \Gamma(D^*, \Omega^{n-1}), \; d\omega = 0 .$$

We can now conclude from (7.6) that

$$\mathrm{Ind}(\gamma; s) = \rho(s), \quad s \in X - U .$$

This reveals that $s \mapsto \mathrm{Ind}(\gamma; s)$ is a compactly supported, locally constant function on $X - U$.

For a given fixed point $s \notin \mathrm{Supp}(b\gamma)$ choose U to be an open neighbourhood of $\mathrm{Supp}(b\gamma)$ with $\bar{U}$ compact and $s \notin \bar{U}$. We can apply the considerations above and conclude that the winding number is constant in a neighbourhood of s and zero outside some compact neighbourhood of $\mathrm{Supp}(b\gamma)$. Q.E.D.

(7.11) COROLLARY. *Let* γ *be a compact n-chain on the oriented smooth manifold* X *and* U *an open subset of* X *containing* $\mathrm{Supp}(b\gamma)$. *The relative de Rham homology class*

$$[\gamma] \in H_n^c(X, U; \mathbf{C})$$

is zero if and only if $\mathrm{Ind}(\gamma; s) = 0$ *for all* $s \in X - U$.

Proof. This is a corollary to the proof of (7.10) rather than the statement (7.10). Anyway, the basic point is Poincaré duality (6.6). Q.E.D.

8. CAUCHY'S RESIDUE THEOREM

We shall consider a smooth map $\gamma: S^{n-1} \to E$ from the oriented $n - 1$ sphere into an oriented n-dimensional real vector space E. For a point s outside $\gamma(S^{n-1})$ pick a closed $(n-1)$-form ω_s on $E - \{s\}$ with $\mathrm{Tr}(\omega_s; s) = 1$ and define the *winding number* of γ with respect to s to be

$$\mathrm{Ind}(\gamma; s) = \int_{S^{n-1}} \gamma^* \omega_s . \tag{8.1}$$

(8.2) CAUCHY'S RESIDUE THEOREM. *Let* $\gamma: S^{n-1} \to X$ *denote a smooth map into an open subset* X *of* E *with* $\mathrm{Ind}(\gamma; z) = 0$ *for all* $z \in E - X$.

For a closed and discrete subset S *of* X *disjoint from* $\gamma(S^{n-1})$ *only finitely many of the numbers* $\mathrm{Ind}(\gamma; s)$, $s \in S$, *are distinct from zero and*

$$\int_{S^{n-1}} \gamma^*\omega = \sum_{s \in S} \mathrm{Ind}(\gamma; s)\mathrm{Tr}(\omega; s)$$

for any closed form ω *on* $X - S$.

Proof. The long exact de Rham homology sequence for the pair $X - S, E$ degenerates into an isomorphism

$$b: H_n^c(E, X-S; \mathbf{C}) \xrightarrow{\sim} H_{n-1}^c(X-S, \mathbf{C}).$$

Let us view γ as a homology class on $X - S$ and introduce the class

$$b^{-1}\gamma \in H_n^c(E, X-S; \mathbf{C}).$$

Let us notice that the winding number (8.1) and (7.8) agree. Thus we conclude from (7.11) that $b^{-1}\gamma$ maps to zero in $H_n^c(E, X; \mathbf{C})$ and consequently that γ is homologous to zero on X. The exact sequence

$$0 \to H_n^c(X, X-S; \mathbf{C}) \xrightarrow{b} H_{n-1}^c(X-S, \mathbf{C}) \to H_{n-1}^c(X, \mathbf{C})$$

allows us to interpret γ as a relative class

$$\gamma \in H_n^c(X, X-S; \mathbf{C}).$$

The class γ can be specified by the formula

$$<b\gamma, \omega> = \int_{S^{n-1}} \gamma^*\omega, \quad \omega \in \Gamma(X-S, \Omega^{n-1}), \ d\omega = 0.$$

From the decomposition (4.9) and excision (4.6) we deduce a canonical isomorphism

$$H_n(X, X-S; \mathbf{C}) \xrightarrow{\sim} \bigoplus_{s \in S} H_n(X, X-\{s\}; \mathbf{C})$$

which allow us to decompose the class γ into a finite sum, compare (7.6),

$$\gamma = \sum_{s \in S} \mathrm{Ind}(\gamma; s)\theta_s.$$

Using the general Stokes formula (5.3) we get that

$$<b\gamma, \omega> = <\gamma, \partial\omega> = \sum \mathrm{Ind}(\gamma; s) <\theta_s, \partial\omega>$$

and the result follows from formula (7.6). Q.E.D.