KNOTTING CODIMENSION 2 SUBMANIFOLDS LOCALLY

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INTRODUCTION

Let L be a connected oriented n-dimensional closed manifold smoothly embedded in a connected oriented $(n+2)$-dimensional closed manifold M, and let K be an oriented n-dimensional smooth knot in the oriented S^{n+2}. Then we consider the connected sum $(M, L) \sharp (S^{n+2}, K)$. In other words, we knot L locally using K. It yields another embedding of L in M; however, it does not always give a new embedding. In fact, the lightbulb theorem says that the connected sum of $(S^2 \times S^1, \{*\} \times S^1)$ with any knot in S^3 is always equivalent to the original embedding. Moreover, by the prime decomposition theorem for knots in 3-manifolds [My], $(S^2 \times S^1, \{*\} \times S^1)$ is essentially the only embedding of a circle with the above property. Litherland [Li] has generalized the lightbulb theorem to the higher dimensional cases. In the appendix of [V], Viro exhibits an example of a 2-knot whose connected sum with the standard projective plane in S^4 does not change the isotopy type of the projective plane. (See also [La].)

The purpose of this paper is to study under what conditions this phenomenon occurs (or does not occur). The first named author [Ms] studied this problem when the codimension is greater than 2.

Put it in another way. Let $\mathcal{K}_n$ be the set of isotopy classes of oriented n-knots diffeomorphic to S^n in the oriented S^{n+2}. The set forms an abelian monoid under connected sum for pairs. Analogously to the inertia group of a manifold, we define

$$I(M, L) = \{(S^{n+2}, K) \in \mathcal{K}_n \mid (M, L) \sharp (S^{n+2}, K) = (M, L)\}$$

where $=$ in the parenthesis indicates that there is an orientation preserving diffeomorphism of pairs. The set forms a submonoid of $\mathcal{K}_n$ and describes the effect of knotting L locally. We are also concerned with the following intermediate submonoid

$$I_0(M, L) = \{(S^{n+2}, K) \in I(M, L) \mid (M, L) \sharp (S^{n+2}, K) \equiv (M, L)\}$$

where $\equiv$ indicates that there is an orientation preserving diffeomorphism of pairs which is concordant to the identity map as a diffeomorphism of the ambient space M.

Our results suggest that $I(M, L)$ and $I_0(M, L)$ depend only on the order of a meridian of L in $\pi_1(M-L)$ or $H_1(M-L;\mathbf{Z})$. Roughly speaking, according as the order is infinite, 1, or p $(1<p<\infty)$, they can be distinguished by (at least) these three types:

Type 1 $I(M, L) = \{0N\}$,

Type 2 $I(M, L) = \mathscr{K}_n$, $\quad I_0(M, L) = \ker\sigma$,

Type 3 $\{0\} \subsetneq I(M, L) \subsetneq \mathscr{K}_n$, $\quad \{0\} \subsetneq I_0(M, L) \subsetneq \ker\sigma$,

(see section 4 for $\sigma(S^{n+2}, K)$).

We refer the reader to 1.1, 2.6, 3.4, 5.1, 5.2, and 5.8 for the precise statement.

This paper consists of five sections. In Section 1, we deduce a necessary condition for $I_0(M, L)$, which is valid for any (M, L). We treat type 1 in Section 2. Type 2 is discussed in Sections 3, 4 and type 3 is discussed in Section 5. We will find that type 3 is closely related to the generalized Smith conjecture.

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§ 1. General remarks on $I_0(M, L)$

It is known (and it is easily verified) that the signature of a Seifert surface of an oriented n-knot K in S^{n+2} is independent of the choice of a Seifert surface; so it is an invariant of the oriented knot K. The invariant is called the signature of the knot K and denoted by $\mathrm{Sign}(S^{n+2}, K)$. We note that $\mathrm{Sign}(S^{n+2}, K)$ is trivially zero unless $n+1 \equiv 0\ (4)$.

As is seen in Section 3, there is a pair (M^{n+2}, L^n) such that $I(M, L) = \mathscr{K}_n$ for any $n \geqslant 3$. In contrast, we can deduce a necessary condition for $I_0(M, L)$ which holds for any pair (M, L).

THEOREM 1.1. *If* $(S^{n+2}, K) \in I_0(M, L)$, *then* $\mathrm{Sign}(S^{n+2}, K) = 0$.

Proof. Let V be a Seifert surface of K. Since $S^{n+2} = \partial D^{n+3}$, we can push the interior of V into the interior of D^{n+3} so that V is transverse to S^{n+2}. This yields an oriented pair (D^{n+3}, V) having (S^{n+2}, K) as the boundary.

The boundary connected sum $(M, L) \times I \natural (D^{n+3}, V)$ gives a cobordism between $(M, L) \# (S^{n+2}, K)$ and (M, L). We note that the ambient space of the cobordism is diffeomorphic to $M \times I$. Since $(S^{n+2}, K) \in I_0(M, L)$, there is an orientation preserving diffeomorphism $f: (M, L) \# (S^{n+2}, K) \to (M, L)$ which is concordant to the identity when regarded as a diffeomorphism of the ambient space M. We paste togethor $(M, L) \# (S^{n+2}, K)$ and (M, L) by f to get an oriented pair of closed manifolds. Since f is concordant to the identity, the resulting ambient space is diffeomorphic to $M \times S^1$. We shall denote by X the resulting oriented closed submanifold of $M \times S^1$.

The additivity property of the signature (see [AS, p. 588]) says that

$$\operatorname{Sign} X = \operatorname{Sign} L \times I + \operatorname{Sign} V = \operatorname{Sign} V,$$

where $\operatorname{Sign} L \times I = 0$ follows easily from the definition of the signature of a manifold with boundary. By the Hirzebruch signature theorem (see [MS, § 19]) we have

$$\operatorname{Sign} X = \mathscr{L}(X)[X]$$

where the right hand side means the Hirzebruch L-class $\mathscr{L}(X)$ of X evaluated on the fundamental class $[X]$ of X. In the sequel we shall show $\mathscr{L}(X)[X] = 0$.

Let $j: X \to M \times S^1$ be the inclusion map. Then it is not difficult to see that

(1.2) $$j_*[X] = [L \times S^1] \quad \text{in} \quad H_{n+1}(M \times S^1; \mathbf{Z})$$

where $[L \times S^1]$ denotes the homology class represented by $L \times S^1$.

Let ν be the normal bundle to X in $M \times S^1$. By the multiplicativity of L-class we have

(1.3) $$\begin{aligned} \mathscr{L}(X) &= \mathscr{L}(\nu)^{-1} j^* \mathscr{L}(M \times S^1) \\ \mathscr{L}(M \times S^1) &= \mathscr{L}(M) \times \mathscr{L}(S^1) = \pi^* \mathscr{L}(M) \end{aligned}$$

where $\pi: M \times S^1 \to M$ is the projection map. Since $\dim \nu = 2$, we have

(1.4) $$\mathscr{L}(\nu) = 1 + p_1(\nu)/3 = 1 + e(\nu)^2/3$$

where p_1 and e denote the first Pontrjagin class and the Euler class respectively.

On the other hand it is known that

(1.5) $$e(\nu) = j^* j_!(1)$$

where $j_! : H^q(X;\mathbf{Z}) \to H^{q+2}(M \times S^1;\mathbf{Z})$ denotes the Gysin homomorphism and $1 \in H^0(X;\mathbf{Z})$ is the unit element. Remember the definition of $j_!$. It is defined so that the following diagram commutes:

$$\begin{array}{ccc} H^q(X;\mathbf{Z}) & \xrightarrow{j_!} & H^{q+2}(M \times S^1;\mathbf{Z}) \\ \downarrow \cap[X] & & \downarrow \cap[M \times S^1] \\ H_{n+1-q}(X;\mathbf{Z}) & \xrightarrow{j_*} & H_{n+1-q}(M \times S^1;\mathbf{Z}) \end{array}$$

where the vertical maps are the Poincaré dualities. It says that

$$j_!(1) \cap [M \times S^1] = j_*[X].$$

This together with (1.2) means that

$$j_!(1) \in \pi^* H^2(M;\mathbf{Z}).$$

Hence it follows from (1.4) and (1.5) that

$$\mathscr{L}(\nu) \in j^*\pi^* H^*(M;Q)$$

and hence

$$\mathscr{L}(X) \in j^*\pi^* H^*(M;Q)$$

by (1.3). This together with (1.2) implies that

$$\mathscr{L}(X)[X] = 0. \qquad \text{Q.E.D.}$$

Theorem 1.1 gives a necessary condition for (S^{n+2}, K) to belong to $I_0(M, L)$. When we consider the converse problem, i.e. the problem to find (S^{n+2}, K) in $I_0(M, L)$, we apply the relative s-cobordism theorem. We shall state it as a lemma for later convenience's sake.

LEMMA 1.6. *Suppose there exists a cobordism (U, Z) between $(M, L) \,\sharp\, (S^{n+2}, K)$ and (M, L) such that*

(1) *Z is diffeomorphic to $L \times I$,*

(2) *the exterior $E(Z)$ of Z is an s-cobordism relative boundary.*

Then $(S^{n+2}, K) \in I_0(M, L)$.

Proof. The relative s-cobordism theorem says that $E(Z)$ is diffeomorphic to $E(L) \times I$ where the diffeomorphism can be taken as the identity on $E(L) \times \{0\}$ and $(\partial E(L)) \times I$. Therefore it extends to a diffeomorphism: $(U, Z) \to (M, L) \times I$ which is the identity on the 0-level. This means that $(S^{n+2}, K) \in I_0(M, L)$. Q.E.D.

§ 2. Type 1 case

In this section we consider the case where a meridian of L^n in M^{n+2} has infinite order in $H_1(M-L;\mathbf{Z})$. We shall denote by $[m]$ the homology class in $H_1(M-L;\mathbf{Z})$ represented by a meridian m of L in M. For a manifold pair (X, Y) of codimension 2 and an epimorphism γ from $\pi_1(X-Y)$ to a finite group, let $(X, Y)_\gamma$ be the branched covering of (X, Y) corresponding to γ. Each knot group $\pi_1(S^{n+2}-K)$ has a natural epimorphism to $\mathbf{Z}_p$ for any positive integer p, and the corresponding p-fold branched cyclic covering of (S^{n+2}, K) is denoted by $(S^{n+2}, K)_p$.

Lemma 2.1. *Suppose* $[m]$ *is of infinite order. Then if* $(S^{n+2}, K) \in I(M, L)$ *then* $(S^{n+2}, K)_p$ *is a homotopy* $(n+2)$*-sphere for any positive integer* p.

Proof. Since $[m]$ represents a nontrivial element in the finitely generated free abelian group $B_1(M-L) \equiv H_1(M-L;\mathbf{Z})/\mathrm{Tor}\, H_1(M-L;\mathbf{Z})$, there is a positive integer r and a primitive element x in $B_1(M-L)$ such that $[m] = rx$ in $B_1(M-L)$. For each positive integer p, ler γ_p be the canonical epimorphism $\pi_1(M-L) \to B_1(M-L) \otimes \mathbf{Z}_{pr}$. Noting the naturality of the homomorphism γ_p, we can see the following:

$$\begin{aligned}(M, L)_{\gamma_p} &= \big((M, L) \sharp (S^{n+2}, K)\big)_{\gamma_p \circ f_*} \\ &= (M, L)_{\gamma_p} \sharp\ d_p (S^{n+2}, K)_p\end{aligned}$$

Here f is a diffeomorphism $(M, L) \sharp (S^{n+2}, K) \to (M, L)$ and d_p is the order of $B_1(M-L) \otimes \mathbf{Z}_{pr}$ divided by p. Hence $H_*((S^{n+2}, K)_p;\mathbf{Z}) \simeq H_*(S^{n+2};\mathbf{Z})$ and $\pi_1\big((S^{n+2}, K)_p\big) \simeq 1$ by the existence of prime decompositions of finitely generated groups into free products [Wg]. Q.E.D.

It is conjectured that those knots which satisfy the conclusion of the above lemma are trivial. In fact, for $n = 1$, it follows from the Smith conjecture [MB]. As a supporting evidence for higher dimensional cases, we have

Lemma. *Suppose that* $(S^{n+2}, K)_p$ *is a homology* $(n+2)$*-sphere for every positive integer* p. *Then the Alexander modules of* K *are trivial.*

Proof. Let $\tilde{E}(K)$ be the infinite cyclic cover of the exterior $E(K)$ of K in S^{n+2}, and let t denote the automorphism of the homology group of $\tilde{E}(K)$ induced by the action of a meridian. Then, by the arguments of [Sm1], we can see that $t^p - 1: H_q(\tilde{E}(K); \mathbf{Z}_r) \to H_q(\tilde{E}(K); \mathbf{Z}_r)$ is an isomorphism for any positive integers p, q, and r. Assume r is prime. Then $H_q(\tilde{E}(K); \mathbf{Z}_r)$ is a finite abelian group, since it is a finitely generated torsion module over the principal ideal domain $\mathbf{Z}_r\langle t\rangle$ (see [Le3, p. 8]). So the automorphism t on $H_q(\tilde{E}(K); \mathbf{Z}_r)$ has a finite order, say d, and we have $t^d - 1 = 0$. Hence $H_q(\tilde{E}(K); \mathbf{Z}_r) = 0$, and by the universal coefficient theorem, the following holds for any prime r and any positive integer q:

$$H_q(\tilde{E}(K); \mathbf{Z}) \otimes \mathbf{Z}_r = 0 \tag{2.3}$$

$$\mathrm{Tor}\,(H_q(\tilde{E}(K); \mathbf{Z}), \mathbf{Z}_r) = 0 \tag{2.4}$$

By (2.4), $H_q(\tilde{E}(K); \mathbf{Z})$ has no nontrivial elements of finite order; so it has a square presentation matrix $M(t)$ as a $Z\langle t\rangle$-module by [Le3, Proposition 3.5]. By (2.3) the q-th Alexander polynomial $\det M_q(t)\,(\in \mathbf{Z}\langle t\rangle)$ is a unit mod. r for any prime r. Hence it is a unit in $\mathbf{Z}\langle t\rangle$, and we have $H_q(\tilde{E}(K); \mathbf{Z}) = 0$ for any positive integer q. Q.E.D.

Thus, as a consequence of Lemmas 2.1 and 2.2 and the results of [Le2] and [T], we have the following:

PROPOSITION 2.5. *Suppose* $[m]$ *is of infinite order. Then any knot in* $I(M, L)$ *has trivial Alexander modules and is null cobordant.*

Hence the only obstruction for a knot (S^{n+2}, K) in $I(M, L)$ to be trivial lies in the knot group $\pi_1(S^{n+2} - K)$. For the special case where $[m]$ generates $H_1(M - L)$, we can apply the result of Maeda [Ma] (cf. [DF]), and obtain the following:

THEOREM 2.6. *Suppose* $n \geqslant 3$ *and* $H_1(M - L)$ *is the infinite cyclic group generated by* $[m]$. *Then* $I(M, L)$ *is trivial.*

Proof. Let (S^{n+2}, K) be a knot in $I(M, L)$. Note that $\pi_1(M - L)$ is isomorphic to the amalgamated free product $\pi_1(M - L) \underset{\langle m\rangle}{*} \pi_1(S^{n+2} - K)$. Then we can conclude $\pi_1(S^{n+2} - K) \simeq \mathbf{Z}$ by the result of [Ma] (cf. [DF]) which asserts the existence of a prime decomposition of a finitely presented group G with $G/[G, G] \simeq \mathbf{Z}$ with respect to such amalgamated free products. Combined with Proposition 2.5, we see $S^{n+2} - K$ is homotopy equivalent to a circle. Hence (S^{n+2}, K) is trivial by [Le1].

§ 3. Type 2 case

In this section and the next section, we treat the case where a meridian of L^n in M^{n+2} is null homotopic in $M - L$. The following lemma follows from [Li, Lemma 1]. We shall give an alternative proof which is interesting by itself (the argument is also given in [Ms, Theorem 4.2]).

LEMMA 3.1. *$I(S^n \times S^2, S^n \times \{*\}) = \mathscr{K}_n$ if $n \geqslant 3$.*

Proof. Let (S^{n+2}, K) be an n-knot and consider $(S^n \times S^2, S^n \times \{*\})$ $\sharp$ (S^{n+2}, K). A subset $S^n \times \{*\}$ $K \cup \{x_0\} \times S^2$ $(x_0 \in S^n)$ is exactly the wedge sum of S^n and S^2. As easily observed the complement of an open regular neighborhood of the subset is contractible and hence diffeomorphic to D^{n+2} as $n + 2 \geqslant 5$. This means that one can express

$$(S^n \times S^2, S^n \times \{*\}) \sharp (S^{n+2}, K) = (S^n \times S^2, S^n \times \{*\}) \sharp \Sigma$$

where Σ is a homotopy $(n+2)$-sphere and the connected sum at the right hand side is done away from the submanifold $S^n \times \{*\}$.

On the other hand the ambient manifold must be diffeomorphic to $S^n \times S^2$ because it is the connected sum of $S^n \times S^2$ with S^{n+2}. These mean that Σ belongs to the inertia group of $S^n \times S^2$. But the group is trivial ([Sc]), so Σ must be the standard sphere. This proves the lemma. Q.E.D.

We shall denote by $<m>$ the class in $\pi_1(M-L)$ represented by a meridian of L in M.

LEMMA 3.2. *Suppose M is spin, L is diffeomorphic to S^n, and $n \geqslant 3$. If $<m> = 1$ for (M, L), then $(M, L) = (S^n \times S^2, S^n \times \{*\}) \sharp M'$ with a closed oriented manifold M' of dimension $n + 2$.*

Proof. Since $<m> = 1$ and $\dim M \geqslant 5$, the meridian m bounds a 2-disk in $M - L$. Therefore $L \vee S^2$ is embedded in M. The normal bundle to L in M is trivial, because it is classified by the Euler class sitting in $H^2(L; \mathbf{Z})$ and $H^2(L; \mathbf{Z}) = 0$ as $L = S^n$ and $n \geqslant 3$. The normal bundle of the embedded S^2 is also trivial, because it is classified by the second Stiefel-Whitney class and it vanishes as M is spin. Hence the closed regular neighborhood of $L \vee S^2$ in M is diffeomorphic to that of $S^n \vee S^2$ naturally embedded in $S^n \times S^2$. In particular its boundary is diffeomorphic to S^{n+1}. This implies the lemma. Q.E.D.

Remark 3.3. A similar argument works even if M is not spin. But this time two cases arise according as the normal bundle of the embedded S^2 is trivial or not. If it is trivial, then the same conclusion as above holds. If it is not trivial, we have

$$(M, L) = (S^n \tilde{\times} S^2, S^n) \# M' .$$

Here $S^n \tilde{\times} S^2$ denotes the total space of the sphere bundle associated with the nontrivial $(n+1)$-dimensional vector bundle over S^2 (note that it is unique as $\pi_1(SO(n+1)) \simeq Z_2$ for $n \geqslant 2$) and the submanifold S^n denotes a fiber.

Combining Lemma 3.1 with 3.2, we obtain

THEOREM 3.4. *Suppose M is spin, L is diffeomorphic to S^n, and $n \geqslant 3$. Then if $<m> = 1$ for (M, L), then $I(M, L) = \mathcal{K}_n$.*

Remark 3.5. If the inertia group $I(S^n \tilde{\times} S^2)$ is trivial, then the same argument as the proof of Lemma 3.1 proves that $I(S^n \tilde{\times} S^2, S^n) = \mathcal{K}_n$ and hence one could drop the spin condition for M by Remark 3.3.

If $L \neq S^n$, then the above argument does not work. For a general L we construct an s-cobordism between pairs $(M, L) \# (S^{n+2}, K)$ and (M, L) and apply lemma 1.6. We denote the set of all null-cobordant n-knots by $\mathcal{K}_n^0$. According to Kervaire [K] (cf. [KW, Chap. IV]) $\mathcal{K}_n = \mathcal{K}_n^0$ if n is even, but $\mathcal{K}_n \neq \mathcal{K}_n^0$ if n is odd.

PROPOSITION 3.6. *Suppose $<m> = 1$ for (M^{n+2}, L^n) and $n \geqslant 3$. Then $I_0(M, L)$ contains $\mathcal{K}_n^0$. In particular, if n is even $\geqslant 4$, then $I_0(M, L) = I(M, L) = \mathcal{K}_n$.*

Proof. Let (S^{n+2}, K) bound a disk pair (D^{n+3}, D), where D is a $(n+1)$-disk. The boundary connected sum $(M, L) \times I \natural (D^{n+3}, D)$ at the 1-level gives a cobordism between (M, L) and $(M, L) \# (S^{n+2}, K)$.

We shall check the conditions (1) and (2) in Lemma 1.6 for this cobordism. First, since D is diffeomorphic to D^{n+1}, $L \times I \natural D$ is diffeomorphic to $L \times I$; so (1) is satisfied. Hence $E(L \times I \natural D)$ gives a cobordism relative boundary between $E(L)$ and $E(L \# K)$. We note that

$$E(L \times I \natural D) = E(L \times I) \cup E(D) \tag{3.7}$$

where $E(L \times I)$ and $E(D)$ are pasted together along $D^{n+1} \times S^1$ embedded in their boundaries. The S^1 factor corresponds to meridians of $L \times I$ and D. Then the van Kampen's theorem says that

$$\pi_1(E(L\times I \natural D)) \simeq \pi_1(E(L\times I)) \underset{<m>}{*} \pi_1(E(D))$$

$$\simeq \pi_1(E(L\times I)) * (\pi_1(E(D))/<m>)$$

where the latter isomorphism is because $<m> = 1$ in $\pi_1(E(L\times I))$ by the assumption. Since $\pi_1(E(D))/<m> \simeq \pi_1(D^{n+3}) \simeq \{1\}$, we have

$$\pi_1(E(L\times I \natural D)) \simeq \pi_1(E(L\times I)) \simeq \pi_1(E(L)). \tag{3.8}$$

Here the inclusion map $i: E(L) = E(L) \times \{0\} \to E(L\times I \natural D)$ induces the isomorphism.

We shall observe that i is a simple homotopy equivalence. For that purpose we consider the lifting of i to the universal covers. Since the map $\pi_1(E(D)) \to \pi_1(E(L\times I \natural D))$ induced by the inclusion map is trivial as observed above, it follows from (3.7) that

$$\tilde{E}(L\times I \natural D) = \tilde{E}(L\times I) \cup E(D) \times \Pi \tag{3.9}$$

where $\Pi = \pi_1(E(L\times I \natural D)) = \pi_1(M-L)$ and $\tilde{E}(L\times I)$ and $E(D) \times \Pi$ are pasted together Π-equivariantly along $D^{n+1} \times S^1 \times \Pi$ embedded in their boundaries. This means that $\tilde{i}_*: H_q(\tilde{E}(L); \mathbf{Z}) \to H_q(\tilde{E}(L\times I \natural D); \mathbf{Z})$ is an isomorphism as $\mathbf{Z}[\Pi]$-modules. Hence $i_*: \pi_q(E(L)) \to \pi_q(E(L\times I \natural D))$ is an isomorphism by Namioka's theorem (see [Wl1, § 4]) and hence i is a homotopy equivalence.

The assumption $<m> = 1$ together with (3.9) tells us that the Whitehead torsion $\tau(i) \in Wh(\Pi)$ of the map i comes from an element of $Wh(1)$ through the map: $Wh(1) \to Wh(\Pi)$ induced from the inclusion $1 \to \Pi$. However $Wh(1) = 0$ and hence $\tau(i) = 0$. This shows that $E(L\times I \natural D)$ is an s-cobordism relative boundary. The proposition then follows from Lemma 1.6. Q.E.D.

Proposition 3.6 gives a complete answer to the case where n is even $\geqslant 4$. It would be interesting to ask if the same conclusion still holds in the case $n = 2$.

In the next section we will improve Proposition 3.6 when n is odd $\geqslant 5$.

§ 4. An improvement

Throughout this section we assume n is odd $\geqslant 5$. Let V^{n+1} be a Seifert surface of an n-knot K in S^{n+2}. The normal bundle to V in S^{n+2} is trivial. We give the stable normal bundle of S^{n+2} a canonical framing so that V can be viewed as a framed manifold.

Remember that $\partial V = K = S^n$. We make V contractible by framed surgery without touching the boundary. As is well known this is always possible in case $\dim V = n + 1$ is odd. But in case $n + 1$ is even, we encounter an obstruction which is detected by

$$\begin{cases} \operatorname{Sign} V \in \mathbf{Z} & \text{if} \quad n + 1 \equiv 0\,(4) \\ c(V) \in \mathbf{Z}/2\mathbf{Z} & \text{if} \quad n + 1 \equiv 2\,(4) \end{cases}$$

where $c(V)$ is the Kervaire invariant of V.

Remark 4.1. Since ∂V is diffeomorphic to S^n, $c(V) = 0$ if $n + 1$ is not of the form $2^k - 2$ ([Br]).

One can see that Seifert surfaces of K are framed cobordant relative boundary to each other. Hence the values $\operatorname{Sign} V$ and $c(V)$ are independent of the choice of V. We set

$$\sigma(S^{n+2}, K) = \begin{cases} \operatorname{Sign} V & \text{if} \quad n + 1 \equiv 0\,(4), \\ c(V) & \text{if} \quad n + 1 = 2^k - 2 \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 4.2. *Suppose* $<m> = 1$ *for* (M^{n+2}, L^n) *and* n *is odd* $\geqslant 5$. *Then* $(S^{n+2}, K) \in I_0(M, L)$ *if* $\sigma(S^{n+2}, K) = 0$. *In particular,* $I_0(M, L) = \mathscr{K}_n$ *if neither* $n + 1 \equiv 0\,(4)$ *nor* $n + 1 = 2^k - 2$ *for some* k.

Combining this with Theorem 1.1, we obtain

COROLLARY 4.3. *Suppose* $<m> = 1$ *for* (M^{n+2}, L^n) *and* $n + 1 \equiv 0\,(4)\,(n \neq 3)$. *Then* $(S^{n+2}, K) \in I_0(M, L)$ *if and only if* $\sigma(S^{n+2}, K) = 0$.

The rest of this section is devoted to the proof of Proposition 4.2. Let K be an n-knot in S^{n+2} such that $\sigma(S^{n+2}, K) = 0$. We shall construct an s-cobordism relative boundary between $E(L \quad K)$ and $E(L)$. The argument is rather more complicated than that of Proposition 3.6. We need some knowledge of surgery theory.

Step 1. Let V^{n+1} be a Seifert surface of K. Push the interior of V into the interior of D^{n+3} to make it transverse to the boundary S^{n+2} of D^{n+3}. We may assume that V is $(n-1)/2$-connected, if necessary, by doing framed surgery of V within D^{n+3}. In fact, this is the method used to prove that any n-knot is concordant to a simple knot (see [KW, Chap. IV]).

In the attempt to make V $(n+1)/2$-connected (and hence V is contractible by the Poincaré duality) by framed surgery of V within D^{n+3}, one encounters an obstruction. Namely a bunch of embedded $(n+1)/2$-spheres in V does not necessarily extend to embedded $(n+3)/2$-disks whose interior lies in $D^{n+3} - V$.

But if we do framed surgery of V at the outside of D^{n+3} without touching boundary, i.e. if we do surgery on framed embeddings

$$(S^{(n+1)/2} \times D^{(n+1)/2} \times D^2, S^{(n+1)/2} \times D^{(n+1)/2} \times \{0\}) \to (D^{n+3}, V),$$

then we can make V $(n+1)/2$-connected because the obstruction is exactly $\sigma(S^{n+2}, K)$ and it vanishes by the assumption. The ambient space is, however, not D^{n+3} any more. We denote by (W, D) the resulting framed oriented pair, where D is diffeomorphic to D^{n+1}.

Step 2. We construct a boundary preserving map h:

$$\big(W; N(D), E(D)\big) \to \big(D^{n+3}; N(D^{n+1}), E(D^{n+1})\big)$$

such that

(4.4) $\quad h|_{\partial W}: \partial W = S^{n+2} \to \partial D^{n+3} = S^{n+2}$ is a homotopy equivalence,

(4.5) $\quad h|_{N(D)}: N(D) \to N(D^{n+1})$ is a diffeomorphism,

where N denotes a closed tubular neighborhood and $D^{n+1} \subset D^{n+3}$ is standardly embedded.

Since D is diffeomorphic to D^{n+1}, there is a diffeomorphism

$$g: (D^{n+1} \times D^2, D^{n+1} \times \{0\}) \to \big(N(D), D\big).$$

Here $D^{n+1} \times D^2$ can be naturally identified with $N(D^{n+1})$; so we define

$$h|_{N(D)} = g^{-1} \tag{4.6}$$

First we extend $h|_{\partial W \cap \partial N(D)} = h|_{\partial E(K)}$ to a map from $E(K)$ to $E(\partial D^{n+1}) = E(S^n)$. The obstruction lies in groups

$$H^{q+1}\big(E(K), \partial E(K); \pi_q(E(S^n))\big).$$

Since $E(S^n)$ is homotopy equivalent to S^1, it suffices to prove

$$H^{q+1}(E(K), \partial E(K); \mathbf{Z}) = 0 \quad \text{for} \quad q = 0, 1. \tag{4.7}$$

On the other hand we have

$$\begin{aligned} H^{q+1}(E(K), \partial E(K); \mathbf{Z}) &\simeq H^{q+1}(S^{n+2}, N(K); \mathbf{Z}) && \text{(by excision)} \\ &\simeq \tilde{H}^q(N(K); \mathbf{Z}) && (\text{if} \quad q+1<n+2) \\ &\simeq \tilde{H}^q(S^n; \mathbf{Z}) \\ &= 0 && (\text{if} \quad q \neq n) \end{aligned}$$

Hence (4.7) is satisfied as $n \geqslant 5$.

Consequently we can extend $h|_{N(D)}$ to a map

$$h|_{N(D)\cup\partial W}\colon (N(D)\cup\partial W, \partial W) \to (N(D^{n+1})\cup\partial D^{n+3}, \partial D^{n+3}).$$

The local degree of $h|_{\partial W}\colon \partial W \to \partial D^{n+3}$ is one because $h|_{\partial W\cap N(D)} = h|_{N(K)}\colon N(K) \to N(S^n)$ is a diffeomorphism by (4.6) and $h(E(K)) \subset E(S^n)$ by the construction. Since ∂W and ∂D^{n+3} are both S^{n+2}, $h|_{\partial W}$ is a homotopy equivalence. Hence (4.4) is satisfied. Moreover (4.5) is also satisfied by (4.6). In the sequel it suffices to extend $h|_{\partial E(D)}$ to a map from $E(D)$ to $E(D^{n+1})$. This time the obstruction lies in groups

$$H^{q+1}(E(D), \partial E(D); \pi_q(E(D^{n+1}))).$$

Since $E(D^{n+1})$ is homotopy equivalent to S^1, it suffices to prove

(4.8) $$H^{q+1}(E(D), \partial E(D); \mathbf{Z}) = 0 \quad \text{for} \quad q = 0, 1.$$

By excision we have

$$H^{q+1}(E(D), \partial E(D); \mathbf{Z}) \simeq H^{q+1}(W, N(D)\cup\partial W; \mathbf{Z}).$$

Remember that W is obtained from D^{n+3} by $(n+1)/2$-surgery. It implies that

$$\tilde{H}^i(W; \mathbf{Z}) = 0 \quad \text{if} \quad i \neq (n+1)/2 + 1.$$

In particular

$$\tilde{H}^i(W; \mathbf{Z}) = 0 \quad \text{for} \quad i \leqslant 3$$

as $n \geqslant 5$. Therefore it follows from the exact sequence of the pair $(W, N(D)\cup\partial W)$ that

$$H^{q+1}(W, N(D)\cup\partial W; \mathbf{Z}) \simeq \tilde{H}^q(N(D)\cup\partial W; \mathbf{Z}) \quad \text{for} \quad q \leqslant 2.$$

Here the Mayer-Vietoris exact sequence of the triad $(N(D)\cup\partial W; N(D), \partial W)$ shows that

$$\tilde{H}^q\,(N(D)\cup\partial W; \mathbf{Z}) = 0 \quad \text{for} \quad q = 0, 1,$$

because $N(D)$ is contractible, $\partial W = S^{n+2}$, and $N(D)\cap\partial W = S^n \times S^1$. Hence (4.8) is satisfied, and we have obtained the desired map h.

Step 3. Since W is framed, the framing of the stable normal bundle $\nu(W)$ of W induces a stable bundle map $b\colon \nu(W) \to \nu(D^{n+3})$ which covers h. The triple $\mathscr{B} = (W, h, b)$ is called a normal map.

The identity map $Id\colon (M, L) \times I \to (M, L) \times I$ gives a normal map where the stable bundle map is also the identity. We shall denote the normal map by $\mathscr{B}_{Id} = ((M, L)\times I, Id, Id)$. The maps h and Id are both diffeomorphisms on $N(D)$ and $N(L\times I)$ respectively; so one can do the boundary connected sum of $\mathscr{B}$ and $\mathscr{B}_{Id}$ at points of K and $L \times \{1\}$. This yields a new normal map $\mathscr{B}_{Id} \natural \mathscr{B} = (M\times I \natural W, Id \natural h, Id \natural b)$. Here we naturally identify the target space $(M, L) \times I \natural (D^{n+3}, D^{n+1})$ with $(M, L) \times I$. Since $Id \# h$ is a diffeomorphism on $N(L\times I \natural D)$, it gives a product structure on $N(L\times I \# D)$. Thus we get a cobordism $E(L\times I \natural D)$ relative boundary between $E(L \# K)$ and $E(L)$.

Step 4. $Id \natural h|_{E(L)}: E(L) \to E(L) \times \{0\}$ (the 0-level) is the identity; so it is a simple homotopy equivalence. We shall observe that $h_1 = Id \natural h|_{E(L\# K)}: E(L \# K) \to E(L) \times \{1\}$ (the 1-level) is also a simple homotopy equivalence.

We have a decomposition

$$E(L \# K) = E(L) \cup E(K)$$

in the same sense as (3.7). Hence, similarly to (3.8) one can see

$$\pi_1(E(L \# K)) \simeq \pi_1(E(L)) \tag{4.9}$$

where the inclusion map induces the isomorphism.

We can view $E(L) \times \{1\}$ as $E(L \# S^n)$ and we also have

$$E(L \# S^n) = E(L) \cup E(S^n)\,.$$

Then the map h_1 can be viewed as the identity on $E(L)$ and h on $E(K)$. This together with (4.9) shows that $h_{1*}: \pi_1(E(L \# K)) \to \pi_1(E(L \# S^n))$ is an isomorphism.

As before we consider the map $\tilde{h}_1: \tilde{E}(L \# K) \to \tilde{E}(L \# S^n)$ lifted to the universal covers. Since $<m> = 1$, we have a diagram

$$\begin{array}{ccccc} \tilde{E}(L \# K) & = & \tilde{E}(L) & \cup & E(K) \times \Pi \\ \downarrow{\scriptstyle \tilde{h}_1} & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle h|_{E(K)}\times Id} \\ \tilde{E}(L \# S^n) & = & \tilde{E}(L) & \cup & E(S^n) \times \Pi\,, \end{array} \tag{4.10}$$

where $\Pi = \pi_1(M-L)$ as before. Since $h|_{E(K)}$ is a homology equivalence, the above diagram tells us that $\tilde{h}_{1*}: H_q(\tilde{E}(L \# K); \mathbf{Z}) \to H_q(\tilde{E}(L \# S^n); \mathbf{Z})$ is an isomorphism as $\mathbf{Z}[\Pi]$-modules. Therefore h_1 is a homotopy equivalence by the same reason as before.

The assumption $<m> = 1$ together with the above diagram tells us that $\tau(h_1) \in Wh(\Pi)$ comes from an element of $Wh(1)$. Hence $\tau(h_1) = 0$ as $Wh(1) = 0$.

Step 5. By step 4 $\bar{h} = Id \natural h|_{E(L\times I\natural D)}: E(L\times I \natural D) \to E(L\times I \natural D^{n+1}) = E(L\times I)$ is a simple homotopy equivalence on the boundary. We convert $\bar{h}$ into a simple homotopy equivalence by surgery without touching the boundary. The obstruction $\sigma(\bar{h})$ lies in an L-group $L_{n+3}(\Pi, 1)$ where 1 denotes the trivial homomorphism from Π to $\mathbf{Z}_2$ (note, since M is oriented and hence so is $E(L\times I)$, the orientation homomorphism: $\Pi = \pi_1(E(L\times I)) \to \mathbf{Z}_2$ is trivial).

We have a diagram similar to (4.10):

$$\begin{array}{ccccc} E(L\times I \natural D) & = & E(L\times I) & \cup & E(D) \\ \bar{h}\downarrow & & \downarrow Id & & \downarrow h \\ E(L\times I \natural D^{n+1}) & = & E(L\times I) & \cup & E(D^{n+1})\,. \end{array}$$

The surgery obstruction $\sigma(h)$ to converting h to a simple homotopy equivalence by surgery without touching the boundary lies in $L_{n+3}(\mathbf{Z}, 1)$ because $\pi_1(E(D^{n+1}))$ is isomorphic to $\mathbf{Z}$. The above diagram together with the assumption $<m> = 1$ tells us that

$$\sigma(\bar{h}) = \beta_*\alpha_*\sigma(h)$$

where $\alpha_*: L_{n+3}(\mathbf{Z}, 1) \to L_{n+3}(1, 1)$ and $\beta_*: L_{n+3}(1, 1) \to L_{n+3}(\Pi, 1)$ are the homomorphisms induced from the trivial homomorphisms $\alpha: \mathbf{Z} \to 1$ and $\beta: 1 \to \Pi$ respectively. It is well-known that

$$L_{n+3}(1, 1) \simeq \begin{cases} \mathbf{Z} & \text{if} \quad n+3 \equiv 0\,(4)\,, \\ \mathbf{Z}_2 & \text{if} \quad n+3 \equiv 2\,(4)\,. \end{cases}$$

As easily observed $\alpha_*\sigma(h)$ is given by

$$\begin{cases} \text{Sign}\, W & \text{if} \quad n+3 \equiv 0\,(4) \\ c(W) & \text{if} \quad n+3 \equiv 2\,(4) \end{cases}$$

through the above isomorphism. Remember that W is framed cobordant to D^{n+3} relative boundary by the construction. Therefore those invariants vanish and hence $\sigma(\bar{h}) = 0$.

Consequently we have obtained a cobordism U' relative boundary between $E(L \# K)$ and $E(L)$ together with a simple homotopy equivalence $F: U' \to E(L\times I)$ which is the identity on the 0-level. Let $i_0: E(L) \to U'$ and $j_0: E(L) \to E(L\times I)$ be the inclusion maps from the 0-level to the cobordisms. Since $F \circ i_0 = j_0 \circ Id$ where $Id: E(L) \to E(L)$ denotes the identity map, we have

$$\tau(F) + F_*\tau(i_0) = \tau(j_0) + j_{0*}\tau(Id)$$

(see [Ml, Lemma 7.8]). Here F, j_0, and Id are all simple homotopy equivalences; so these Whitehead torsions vanish. Hence it follows that $\tau(i_0) = 0$, because $F_*: Wh(\pi_1(U')) \to Wh(\pi_1(E(L \times I))$ is an isomorphism. This means that U' is an s-cobordism. Therefore $(S^{n+2}, K) \in I_0(M, L)$ by Lemma 1.6. Q.E.D.

§ 5. TYPE 3 CASE

In this section we treat the case where $<m>$ or $[m]$ is of order p (p is not necessarily a prime number). We begin with

LEMMA 5.1. *Suppose* $[m]$ *is of order* p. *Then if* $(S^{n+2}, K) \in I(M, L)$, *then* $(S^{n+2}, K)_p$ *is a homotopy* $(n+2)$-*sphere.*

Proof. Let r be the order of Tor $H_1(M-L; \mathbf{Z})$, and let γ be the canonical epimorphism $\pi_1(M-L) \to H_1(M-L; \mathbf{Z}) \otimes \mathbf{Z}_r$. Since the order of $\gamma(<m>)$ is p, we obtain the desired result by an argument similar to the proof of Lemma 2.1. Q.E.D.

If $p \geqslant 2$, there are infinitely many knots (S^{n+2}, K) such that $(S^{n+2}, K)_p$ is not a homotopy $(n+2)$-sphere; so Lemma 5.1 shows that $I(M, L) \subsetneqq \mathscr{K}_n$ for such (M, L).

The rest of this section is devoted to looking for a non-trivial knot in $I(M, L)$ or $I_0(M, L)$. We will extend Proposition 3.6 and 4.2 to the case where $<m>$ is of order p. Lemma 5.1 reminds us of counterexamples to the generalized Smith conjecture.

Let (S^{n+2}, K) be an n-knot which bounds a disk pair (D^{n+3}, D) such that $(D^{n+3}, D)_p$ is a homotopy $(n+3)$-disk. Since $(S^{n+2}, K)_p$ is the boundary of $(D^{n+3}, D)_p$, $(S^{n+2}, K)_p$ is a homotopy $(n+2)$-sphere. If $n + 3 \geqslant 5$, then $(D^{n+3}, D)_p$ is diffeomorphic to D^{n+3} and hence $(S^{n+2}, K)_p$ is diffeomorphic to S^{n+2}.

The p-fold branched cyclic covering $(D^{n+3}, D)_p$ supports a $\mathbf{Z}_p$-action with the branch set D as the fixed point set. Let $E(D)_p$ be the exterior of D in $(D^{n+3}, D)_p$ and let $\rho: S^1 \to E(D)_p$ be an equivariant embedding of a meridian of D in $E(D)_p$, where the standard free $\mathbf{Z}_p$-action is considered on S^1. Since ρ is a homology equivalence and equivariant, the Whitehead torsion of ρ is defined in $Wh(\mathbf{Z}_p)$. Clearly it is independent of the choice of ρ; so we shall denote it by $\tau_p(D^{n+3}, D)$.

The following theorem is an extension of Proposition 3.6.

THEOREM 5.2. *Suppose* $<m>$ *is of order* p *(*p *may be equal to* 1*) for* (M^{n+2}, L^n) *and* $n \geqslant 4$. *Then* $(S^{n+2}, K) \in I_0(M, L)$ *if it bounds a disk pair* (D^{n+3}, D) *such that*

(1) $(D^{n+3}, D)_p$ *is diffeomorphic to* D^{n+3},

(2) $\mu_* \tau_p(D^{n+3}, D) = 0$,

where $\mu_*: Wh(\mathbf{Z}_p) \to Wh(\pi_1(M-L))$ *is the homomorphism induced from a homomorphism* $\mu: \mathbf{Z}_p \to \pi_1(M-L)$ *sending a generator of* $\mathbf{Z}_p$ *to* $<m> \in \pi_1(M-L)$.

Remark 5.3. (1) For each p, there are infinitely many n-knots satisfying the conditions (1) and (2) in Theorem 5.2. For example the $\mathbf{Z}_p$-orbit spaces of Sumners' knots [R, p. 347] (which are counterexamples to the generalized Smith conjecture) are the desired knots. In fact, $\tau_p(D^{n+3}, D) = 0$ for them.

(2) If $p = 1, 2, 3, 4$, or 6, then $Wh(\mathbf{Z}_p) = 0$. Hence the condition (2) of Theorem 5.2 is trivially satisfied in these cases.

Proof of Theorem 5.2. We shall observe that the proof of Proposition 3.6 works with a little modification. As before $E(L \times I \natural D)$ can be viewed as a cobordism relative boundary between $E(L)$ and $E(L \sharp K)$. We shall check that this is an s-cobordism.

The condition (1) implies that

$$(5.4) \qquad \pi_1(E(D))/<m^p> \simeq \mathbf{Z}_p$$

where a meridian of D in D^{n+3} is also denoted by m. Hence it follows from the decomposition (3.7) that

$$\begin{aligned}(5.5) \qquad \pi_1(E(L \times I \natural D)) &\simeq \pi_1(E(L \times I)) \underset{<m>}{*} \pi_1(E(D)) \\ &\simeq \pi_1(E(L \times I)) \underset{\mathbf{Z}_p}{*} \pi_1(E(D))/<m^p> \\ &\quad \text{(as } <m> \text{ is of order } p \text{ in } \pi_1(E(L \times I))) \\ &\simeq \pi_1(E(L \times I)) \qquad \text{(by (5.4))}\end{aligned}$$

This implies that the inclusion map $i: E(L) = E(L) \times \{0\} \to E(L \times I \natural D)$ induces an isomorphism $\pi_1(E(L)) \to \pi_1(E(L \times I \natural D))$.

We consider the map $\tilde{i}: \tilde{E}(L) \to \tilde{E}(L \times I \natural D)$ lifted to the universal cover. Let $q: \tilde{E}(L \times I \natural D) \to E(L \times I \natural D)$ be the covering projection map. By (5.5) $q^{-1}(E(L \times I))$ is exactly the universal cover $\tilde{E}(L \times I)$. As for $q^{-1}(E(D))$ we need a little consideration. The above observation (5.5) shows that the image of $j_*: \pi_1(E(D)) \to \pi_1(E(L \times I \natural D))$ is isomorphic to $\mathbf{Z}_p$, where j is the inclusion map. We shall identify $j_*\pi_1(E(D))$ with $\mathbf{Z}_p$. Remember that $\mathbf{Z}_p$ acts freely on $E(D)_p$ as covering transformations.

Claim 5.6. $q^{-1}(E(D)) = E(D)_p \underset{\mathbf{Z}_p}{\times} \Pi$, where the right hand side denotes the orbit space of $E(D)_p \times \Pi$ by the diagonal $\mathbf{Z}_p$-action defined by $s \cdot (x, g) = (xs^{-1}, sg)$ for $s \in \mathbf{Z}_p$, $x \in E(D)_p$, and $g \in \Pi$.

Proof. The Π-covering $q^{-1}(E(D)) \to E(D)$ is classified by the map: $E(D) \to B\Pi$ induced from the homomorphism $j_*: \pi_1(E(D)) \to \Pi = \pi_1(E(L\times I \natural D))$. Here j_* factors through the inclusion $k: \mathbf{Z}_p \to \Pi$:

$$\begin{tikzcd} \pi_1(E(D)) \arrow[rr, "j_*"] \arrow[dr, "\ell"'] & & \Pi \\ & \mathbf{Z}_p \arrow[ur, hook, "k"'] & \end{tikzcd}$$

The pullback of the universal Π-bundle $E\Pi \to B\Pi$ by k is of the form $E\mathbf{Z}_p \underset{\mathbf{Z}_p}{\times} \Pi \to B\mathbf{Z}_p$. In fact, since $E\mathbf{Z}_p = E\Pi$, the map $(u, g) \to ug$ $(u \in E\mathbf{Z}_p, g \in \Pi)$ is defined from $E\mathbf{Z}_p \underset{\mathbf{Z}_p}{\times} \Pi$ to $E\Pi$. The map induces a Π-bundle map from $E\mathbf{Z}_p \underset{\mathbf{Z}_p}{\times} \Pi \to B\Pi$ to $E\Pi \to B\Pi$. On the other hand the covering induced from the homomorphism $\ell: \pi_1(E(D)) \to \mathbf{Z}_p$ is exactly the $\mathbf{Z}_p$-covering $E(D)_p \to E(D)$. These prove the claim.

Consequently we have a decomposition

$$(5.7) \qquad \tilde{E}(L\times I \natural D) = \tilde{E}(L\times I) \cup E(D)_p \underset{\mathbf{Z}_p}{\times} \Pi,$$

where $\tilde{E}(L\times I)$ and $E(D)_p \underset{\mathbf{Z}_p}{\times} \Pi$ are pasted together along $D^n \times S^1 \underset{\mathbf{Z}_p}{\times} \Pi$ equivariantly embedded in their boundaries. The condition (1) means that $E(D)_p$ is a homology circle. This together with (5.7) tells us that $\tilde{i}: \tilde{E}(L\times I) \to \tilde{E}(L\times I \natural D)$ induces an isomorphism on homology as $\mathbf{Z}[\Pi]$-modules. Hence i is a homotopy equivalence.

The decomposition (5.7) also tells us that

$$\tau(i) = \mu_*\tau_p(D^{n+3}, D) \quad \text{up to sign}.$$

Hence $\tau(i) = 0$ by the condition (2). Therefore $E(L\times I \natural D)$ is an s-cobordism relative boundary. The theorem then follows from Lemma 1.6. Q.E.D.

A torsion $\tau_p(S^{n+2}, K)$ is defined similarly to $\tau_p(D^{n+3}, D)$ if $(S^{n+2}, K)_p$ is a homotopy $(n+2)$-sphere. The following theorem is an extension of Proposition 4.2.

THEOREM 5.8. *Suppose* $<m>$ *is of order* p *(*p *may be equal to* 1*) for* (M^{n+2}, L^n) *and* $n \geqslant 4$. *Let* $a_{n,p} = 2$ *if* $n \equiv 0\,(4)$ *and* p *is even, and let* $a_{n,p} = 1$ *otherwise. Then* $a_{n,p}(S^{n+2}, K) \in I_0(M, L)$ *if*

(1) $\sigma(S^{n+2}, K) = 0$ *in case* n *is odd.*

(2) $(S^{n+2}, K)_p$ *is a homotopy* $(n+2)$*-sphere,*

(3) $a_{n,p}\mu_*\tau_p(S^{n+2}, K) = 0$

where μ_* *is the same as in Theorem 5.2.*

Proof. The argument developed in Steps 1, 2, and 3 of the proof of Proposition 4.2 still works. Step 4 needs a little modification. Instead of (4.10) we have

$$\begin{array}{ccccc} \tilde{E}(L \# K) & = & \tilde{E}(L) \cup E(K)_p \underset{\mathbf{Z}_p}{\times} \Pi \\ \downarrow{\scriptstyle \tilde{h}_1} & & \downarrow{\scriptstyle Id} \qquad\qquad \downarrow{\scriptstyle h_p \underset{\mathbf{Z}_p}{\times} Id} \\ \tilde{E}(L \# S^n) & = & \tilde{E}(L) \cup E(S^n)_p \underset{\mathbf{Z}_p}{\times} \Pi \end{array} \quad , \tag{5.9}$$

(see (5.7)) where $h_p\colon E(K)_p \to E(S^n)_p$ denotes the lifting of h to the $\mathbf{Z}_p$-covers. Since h_p is a homology equivalence, the above diagram tells us that $\tilde{h}_1$ is a homotopy equivalence.

It also tells us that

$$\tau(h_1) = -\mu_*\tau_p(S^{n+2}, K),$$

which vanishes by the condition (3). Hence $h_1\colon E(L \# K) \to E(L \# S^n)$ is a simple homotopy equivalence.

Step 5 also needs some modification. We need to replace α and β by the canonical epimorphism $\gamma\colon \mathbf{Z} \to \mathbf{Z}_p$ and $\mu\colon \mathbf{Z}_p \to \Pi$ respectively. Then we have

$$\sigma(\bar{h}) = \mu_*\gamma_*\sigma(h).$$

Here we distinguish three cases to observe the value $\sigma(\bar{h})$.

Case 1. The case where n is odd. In this case the trivial homomorphism $\alpha\colon \mathbf{Z} \to 1$ induces an isomorphism $L_{n+3}(\mathbf{Z}, 1) \to L_{n+3}(1, 1)$ ([Wl1, 13A.8]). As observed in Step 5 of the proof of Proposition 4.2, $\alpha_*(\sigma(h))$ vanishes. Hence $\sigma(h) = 0$, so $\sigma(\bar{h}) = 0$.

Case 2. The case where $n \equiv 2\,(4)$ or p is odd. According to Wall [Wl2] or Bak [Ba], $L_{n+3}(\mathbf{Z}_p, 1) = 0$ in this case. Since $\gamma_*\sigma(h)$ lies in $L_{n+3}(\mathbf{Z}_p, 1)$, $\gamma_*\sigma(h) = 0$ and hence $\sigma(\bar{h}) = 0$.

Case 3. The case where $n \equiv 0\,(4)$ and p is even. In this case $L_{n+3}(\mathbf{Z}_p, 1) \simeq \mathbf{Z}_2$. Since the value $\gamma_* \sigma(h) \in L_{n+3}(\mathbf{Z}_p, 1)$ is additive with respect to connected sum, it necessarily vanishes for $(S^{n+2}, K) \sharp (S^{n+2}, K)$.

The rest of the argument is the same as that in Step 5. This proves the theorem. Q.E.D.

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