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KNOTTING CODIMENSION 2 SUBMANIFOLDS LOCALLY

by Mikiya Masuda and Makoto Sakuma

INTRODUCTION

Let L be a connected oriented n-dimensional closed manifold smoothly embedded in a connected oriented (n+2)-dimensional closed manifold M, and let K be an oriented n-dimensional smooth knot in the oriented S^{n+2} . Then we consider the connected sum $(M, L) \notin (S^{n+2}, K)$. In other words, we knot L locally using K. It yields another embedding of L in M; however, it does not always give a new embedding. In fact, the lightbulb theorem says that the connected sum of $(S^2 \times S^1, \{*\} \times S^1)$ with any knot in S^3 is always equivalent to the original embedding. Moreover, by the prime decomposition theorem for knots in 3-manifolds [My], $(S^2 \times S^1, \{*\} \times S^1)$ is essentially the only embedding of a circle with the above property. Litherland [Li] has generalized the lightbulb theorem to the higher dimensional cases. In the appendix of [V], Viro exhibits an example of a 2-knot whose connected sum with the standard projective plane in S⁴ does not change the isotopy type of the projective plane. (See also [La].)

The purpose of this paper is to study under what conditions this phenomenon occurs (or does not occur). The first named author [Ms] studied this problem when the codimension is greater than 2.

Put it in another way. Let \mathscr{K}_n be the set of isotopy classes of oriented *n*-knots diffeomorphic to S^n in the oriented S^{n+2} . The set forms an abelian monoid under connected sum for pairs. Analogously to the inertia group of a manifold, we define

$$I(M, L) = \{ (S^{n+2}, K) \in \mathcal{K}_n \mid (M, L) \ \# (S^{n+2}, K) = (M, L) \}$$

where = in the parenthesis indicates that there is an orientation preserving diffeomorphism of pairs. The set forms a submonoid of \mathscr{K}_n and describes the effect of knotting L locally. We are also concerned with the following intermediate submonoid

$$I_0(M, L) = \{ (S^{n+2}, K) \in I(M, L) \mid (M, L) \ \# (S^{n+2}, K) \equiv (M, L) \}$$

where \equiv indicates that there is an orientation preserving diffeomorphism of pairs which is concordant to the identity map as a diffeomorphism of the ambient space M.

Our results suggest that I(M, L) and $I_0(M, L)$ depend only on the order of a meridian of L in $\pi_1(M-L)$ or $H_1(M-L; \mathbb{Z})$. Roughly speaking, according as the order is infinite, 1, or p(1 , they can be distinguishedby (at least) these three types:

Type 1 $I(M, L) = \{0N\},\$

Type 2 $I(M, L) = \mathscr{K}_n, \quad I_0(M, L) = \ker \sigma,$

 $Type \ 3 \quad \{0\} \underset{\neq}{\subset} I(M, L) \underset{\neq}{\subset} \mathscr{K}_n, \quad \{0\} \underset{\neq}{\subset} I_0(M, L) \underset{\neq}{\subset} \ker \sigma,$

(see section 4 for $\sigma(S^{n+2}, K)$).

We refer the reader to 1.1, 2.6, 3.4, 5.1, 5.2, and 5.8 for the precise statement.

This paper consists of five sections. In Section 1, we deduce a necessary condition for $I_0(M, L)$, which is valid for any (M, L). We treat type 1 in Section 2. Type 2 is discussed in Sections 3, 4 and type 3 is discussed in Section 5. We will find that type 3 is closely related to the generalized Smith conjecture.

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§ 1. General remarks on $I_0(M, L)$

It is known (and it is easily verified) that the signature of a Seifert surface of an oriented *n*-knot K in S^{n+2} is independent of the choice of a Seifert surface; so it is an invariant of the oriented knot K. The invariant is called the signature of the knot K and denoted by Sign (S^{n+2}, K) . We note that Sign (S^{n+2}, K) is trivially zero unless $n + 1 \equiv 0$ (4).

As is seen in Section 3, there is a pair (M^{n+2}, L^n) such that $I(M, L) = \mathscr{K}_n$ for any $n \ge 3$. In contrast, we can deduce a necessary condition for $I_0(M, L)$ which holds for any pair (M, L).

THEOREM 1.1. If $(S^{n+2}, K) \in I_0(M, L)$, then $Sign(S^{n+2}, K) = 0$.

Proof. Let V be a Seifert surface of K. Since $S^{n+2} = \partial D^{n+3}$, we can push the interior of V into the interior of D^{n+3} so that V is transverse to S^{n+2} . This yields an oriented pair (D^{n+3}, V) having (S^{n+2}, K) as the boundary.

The boundary connected sum $(M, L) \times I \nmid (D^{n+3}, V)$ gives a cobordism between $(M, L) \notin (S^{n+2}, K)$ and (M, L). We note that the ambient space of the cobordism is diffeomorphic to $M \times I$. Since $(S^{n+2}, K) \in I_0(M, L)$, there is an orientation preserving diffeomorphism $f: (M, L) \notin (S^{n+2}, K) \to (M, L)$ which is concordant to the identity when regarded as a diffeomorphism of the ambient space M. We paste togethor $(M, L) # (S^{n+2}, K)$ and (M, L) by f to get an oriented pair of closed manifolds. Since f is concordant to the identity, the resulting ambient space is diffeomorphic to $M \times S^1$. We shall denote by X the resulting oriented closed submanifold of $M \times S^1$.

The additivity property of the signature (see [AS, p. 588]) says that

$$\operatorname{Sign} X = \operatorname{Sign} L \times I + \operatorname{Sign} V = \operatorname{Sign} V$$
,

where Sign $L \times I = 0$ follows easily from the definition of the signature of a manifold with boundary. By the Hirzebruch signature theorem (see $[MS, \S 19])$ we have

$$\operatorname{Sign} X = \mathscr{L}(X)[X]$$

where the right hand side means the Hirzebruch L-class $\mathscr{L}(X)$ of X evaluated on the fundamental class [X] of X. In the sequel we shall show $\mathscr{L}(X)[X] = 0.$

Let $j: X \to M \times S^1$ be the inclusion map. Then it is not difficult to see that

(1.2)
$$j_*[X] = [L \times S^1]$$
 in $H_{n+1}(M \times S^1; \mathbb{Z})$

where $[L \times S^1]$ denotes the homology class represented by $L \times S^1$.

Let v be the normal bundle to X in $M \times S^1$. By the multiplicativity of *L*-class we have

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$$\mathscr{L}(X) = \mathscr{L}(v)^{-1} j^* \mathscr{L}(M \times S^1)$$

$$\mathscr{L}(M \times S^1) = \mathscr{L}(M) \times \mathscr{L}(S^1) = \pi^* \mathscr{L}(M)$$

where $\pi: M \times S^1 \to M$ is the projection map. Since dim v = 2, we have

(1.4)
$$\mathscr{L}(v) = 1 + p_1(v)/3 = 1 + e(v)^2/3$$

where p_1 and e denote the first Pontrjagin class and the Euler class respectively.

On the other hand it is known that

 $(0(\mathbf{x}))$

(1.5)
$$e(v) = j^* j_1(1)$$

where $j_1 : H^q(X; \mathbb{Z}) \to H^{q+2}(M \times S^1; \mathbb{Z})$ denotes the Gysin homomorphism and $1 \in H^0(X; \mathbb{Z})$ is the unit element. Remember the definition of j_1 . It is defined so that the following diagram commutes:

$$H^{q}(X; \mathbf{Z}) \xrightarrow{j_{1}} H^{q+2}(M \times S^{1}; \mathbf{Z})$$

$$\downarrow \cap [X] \qquad \qquad \downarrow \cap [M \times S^{1}]$$

$$H_{n+1-q}(X; \mathbf{Z}) \xrightarrow{j_{*}} H_{n+1-q}(M \times S^{1}; \mathbf{Z})$$

where the vertical maps are the Poincaré dualities. It says that

 $j_!(1) \cap [M \times S^1] = j_*[X] \,.$

This together with (1.2) means that

$$j_{1}(1) \in \pi^{*}H^{2}(M; \mathbb{Z})$$
.

Hence it follows from (1.4) and (1.5) that

 $\mathscr{L}(\mathbf{v}) \in j^*\pi^*H^*(M;Q)$

and hence

 $\mathscr{L}(X) \in j^* \pi^* H^*(M; Q)$

by (1.3). This together with (1.2) implies that

 $\mathscr{L}(X)[X] = 0. \qquad \text{Q.E.D.}$

Theorem 1.1 gives a necessary condition for (S^{n+2}, K) to belong to $I_0(M, L)$. When we consider the converse problem, i.e. the problem to find (S^{n+2}, K) in $I_0(M, L)$, we apply the relative s-cobordism theorem. We shall state it as a lemma for later convenience's sake.

LEMMA 1.6. Suppose there exists a cobordism (U, Z) between (M, L)# (S^{n+2}, K) and (M, L) such that

(1) Z is diffeomorphic to $L \times I$,

(2) the exterior E(Z) of Z is an s-cobordism relative boundary. Then $(S^{n+2}, K) \in I_0(M, L)$.

Proof. The relative s-cobordism theorem says that E(Z) is diffeomorphic to $E(L) \times I$ where the diffeomorphism can be taken as the identity on $E(L) \times \{0\}$ and $(\partial E(L)) \times I$. Therefore it extends to a diffeomorphism: $(U, Z) \rightarrow (M, L) \times I$ which is the identity on the 0-level. This means that $(S^{n+2}, K) \in I_0(M, L)$. Q.E.D.

§ 2. Type 1 case

In this section we consider the case where a meridian of L^n in M^{n+2} has infinite order in $H_1(M-L; \mathbb{Z})$. We shall denote by [m] the homology class in $H_1(M-L; \mathbb{Z})$ represented by a meridian m of L in M. For a manifold pair (X, Y) of codimension 2 and an epimorphism γ from $\pi_1(X-Y)$ to a finite group, let $(X, Y)_{\gamma}$ be the branched covering of (X, Y) corresponding to γ . Each knot group $\pi_1(S^{n+2}-K)$ has a natural epimorphism to \mathbb{Z}_p for any positive integer p, and the corresponding p-fold branched cyclic covering of (S^{n+2}, K) is denoted by $(S^{n+2}, K)_p$.

LEMMA 2.1. Suppose [m] is of infinite order. Then if $(S^{n+2}, K) \in I(M, L)$ then $(S^{n+2}, K)_p$ is a homotopy (n+2)-sphere for any positive integer p.

Proof. Since [m] represents a nontrivial element in the finitely generated free abelian group $B_1(M-L) \equiv H_1(M-L; \mathbb{Z})/\text{Tor } H_1(M-L; \mathbb{Z})$, there is a positive integer r and a primitive element x in $B_1(M-L)$ such that [m] = rxin $B_1(M-L)$. For each positive integer p, ler γ_p be the canonical epimorphism $\pi_1(M-L) \to B_1(M-L) \otimes \mathbb{Z}_{pr}$. Noting the naturality of the homomorphism γ_p , we can see the following:

$$(M, L)_{\gamma_p} = ((M, L) \sharp (S^{n+2}, K))_{\gamma_p \circ f_*}$$
$$= (M, L)_{\gamma_p} \sharp d_p (S^{n+2}, K)_p$$

Here f is a diffeomorphism $(M, L) \notin (S^{n+2}, K) \to (M, L)$ and d_p is the order of $B_1(M-L) \otimes \mathbb{Z}_{pr}$ divided by p. Hence $H_*((S^{n+2}, K)_p; \mathbb{Z}) \simeq H_*(S^{n+2}; \mathbb{Z})$ and $\pi_1((S^{n+2}, K)_p) \simeq 1$ by the existence of prime decompositions of finitely generated groups into free products [Wg]. Q.E.D.

It is conjectured that those knots which satisfy the conclusion of the above lemma are trivial. In fact, for n = 1, it follows from the Smith conjecture [MB]. As a supporting evidence for higher dimensional cases, we have

LEMMA. Suppose that $(S^{n+2}, K)_p$ is a homology (n+2)-sphere for every positive integer p. Then the Alexander modules of K are trivial.

Proof. Let $\tilde{E}(K)$ be the infinite cyclic cover of the exterior E(K) of K in S^{n+2} , and let t denote the automorphism of the homology group of $\tilde{E}(K)$ induced by the action of a meridian. Then, by the arguments of [Sm1],

we can see that $t^p - 1: H_q(\tilde{E}(K); \mathbb{Z}_r) \to H_q(\tilde{E}(K); \mathbb{Z}_r)$ is an isomorphism for any positive integers p, q, and r. Assume r is prime. Then $H_q(\tilde{E}(K); \mathbb{Z}_r)$ is a finite abelian group, since it is a finitely generated torsion module over the principal ideal domain $\mathbb{Z}_r < t >$ (see [Le3, p. 8]). So the automorphism t on $H_q(\tilde{E}(K); \mathbb{Z}_r)$ has a finite order, say d, and we have $t^d - 1 = 0$. Hence $H_q(\tilde{E}(K); \mathbb{Z}_r) = 0$, and by the universal coefficient theorem, the following holds for any prime r and any positive integer q:

(2.3)
$$H_q(\tilde{E}(K); \mathbf{Z}) \otimes \mathbf{Z}_r = 0$$

(2.4)
$$\operatorname{Tor}\left(H_q(\widetilde{E}(K);\mathbf{Z}),\mathbf{Z}_r\right) = 0$$

By (2.4), $H_q(\tilde{E}(K); \mathbb{Z})$ has no nontrivial elements of finite order; so it has a square presentation matrix M(t) as a Z < t >-module by [Le3, Proposition 3.5]. By (2.3) the q-th Alexander polynomial det $M_q(t) (\in \mathbb{Z} < t >)$ is a unit mod. r for any prime r. Hence it is a unit in $\mathbb{Z} < t >$, and we have $H_q(\tilde{E}(K); \mathbb{Z}) = 0$ for any positive integer q. Q.E.D.

Thus, as a consequence of Lemmas 2.1 and 2.2 and the results of [Le2] and [T], we have the following:

PROPOSITION 2.5. Suppose [m] is of infinite order. Then any knot in I(M, L) has trivial Alexander modules and is null cobordant.

Hence the only obstruction for a knot (S^{n+2}, K) in I(M, L) to be trivial lies in the knot group $\pi_1(S^{n+2}-K)$. For the special case where [m] generates $H_1(M-L)$, we can apply the result of Maeda [Ma] (cf. [DF]), and obtain the following:

THEOREM 2.6. Suppose $n \ge 3$ and $H_1(M-L)$ is the infinite cyclic group generated by [m]. Then I(M, L) is trivial.

Proof. Let (S^{n+2}, K) be a knot in I(M, L). Note that $\pi_1(M-L)$ is isomorphic to the amalgamated free product $\pi_1(M-L) * \pi_1(S^{n+2}-K)$. Then we can conclude $\pi_1(S^{n+2}-K) \simeq \mathbb{Z}$ by the result of [Ma] (cf. [DF]) which asserts the existence of a prime decomposition of a finitely presented group G with $G/[G, G] \simeq \mathbb{Z}$ with respect to such amalgamated free products. Combined with Proposition 2.5, we see $S^{n+2} - K$ is homotopy equivalent to a circle. Hence (S^{n+2}, K) is trivial by [Le1].

§ 3. Type 2 case

In this section and the next section, we treat the case where a meridian of L^n in M^{n+2} is null homotopic in M - L. The following lemma follows from [Li, Lemma 1]. We shall give an alternative proof which is interesting by itself (the argument is also given in [Ms, Theorem 4.2]).

LEMMA 3.1. $I(S^n \times S^2, S^n \times \{*\}) = \mathscr{K}_n \text{ if } n \ge 3.$

Proof. Let (S^{n+2}, K) be an *n*-knot and consider $(S^n \times S^2, S^n \times \{*\})$ $\# (S^{n+2}, K)$. A subset $S^n \times \{*\}$ $K \cup \{x_0\} \times S^2$ $(x_0 \in S^n)$ is exactly the wedge sum of S^n and S^2 . As easily observed the complement of an open regular neighborhood of the subset is contractible and hence diffeomorphic to D^{n+2} as $n + 2 \ge 5$. This means that one can express

$$(S^n \times S^2, S^n \times \{*\}) \ \# (S^{n+2}, K) = (S^n \times S^2, S^n \times \{*\}) \ \# \Sigma$$

where Σ is a homotopy (n+2)-sphere and the connected sum at the right hand side is done away from the submanifold $S^n \times \{*\}$.

On the other hand the ambient manifold must be diffeomorphic to $S^n \times S^2$ because it is the connected sum of $S^n \times S^2$ with S^{n+2} . These mean that Σ belongs to the inertia group of $S^n \times S^2$. But the group is trivial ([Sc]), so Σ must be the standard sphere. This proves the lemma. Q.E.D.

We shall denote by $\langle m \rangle$ the class in $\pi_1(M-L)$ represented by a meridian of L in M.

LEMMA 3.2. Suppose M is spin, L is diffeomorphic to S^n , and $n \ge 3$. If $\langle m \rangle = 1$ for (M, L), then $(M, L) = (S^n \times S^2, S^n \times \{*\}) \# M'$ with a closed oriented manifold M' of dimension n + 2.

Proof. Since $\langle m \rangle = 1$ and dim $M \ge 5$, the meridian *m* bounds a 2-disk in M - L. Therefore $L \lor S^2$ is embedded in *M*. The normal bundle to *L* in *M* is trivial, because it is classified by the Euler class sitting in $H^2(L; \mathbb{Z})$ and $H^2(L; \mathbb{Z}) = 0$ as $L = S^n$ and $n \ge 3$. The normal bundle of the embedded S^2 is also trivial, because it is classified by the second Stiefel-Whitney class and it vanishes as *M* is spin. Hence the closed regular neighborhood of $L \lor S^2$ in *M* is diffeomorphic to that of $S^n \lor S^2$ naturally embedded in $S^n \times S^2$. In particular its boundary is diffeomorphic to S^{n+1} . This implies the lemma. Q.E.D.

Remark 3.3. A similar argument works even if M is not spin. But this time two cases arise according as the normal bundle of the embedded S^2 is trivial or not. If it is trivial, then the same conclusion as above holds. If it is not trivial, we have

$$(M, L) = (S^n \tilde{\times} S^2, S^n) \# M'$$

Here $S^n \times S^2$ denotes the total space of the sphere bundle associated with the nontrivial (n+1)-dimensional vector bundle over S^2 (note that it is unique as $\pi_1(SO(n+1)) \simeq Z_2$ for $n \ge 2$) and the submanifold S^n denotes a fiber.

Combining Lemma 3.1 with 3.2, we obtain

THEOREM 3.4. Suppose M is spin, L is diffeomorphic to S^n , and $n \ge 3$. Then if $\langle m \rangle = 1$ for (M, L), then $I(M, L) = \mathcal{K}_n$.

Remark 3.5. If the inertia group $I(S^n \times S^2)$ is trivial, then the same argument as the proof of Lemma 3.1 proves that $I(S^n \times S^2, S^n) = \mathscr{K}_n$ and hence one could drop the spin condition for M by Remark 3.3.

If $L \neq S^n$, then the above argument does not work. For a general L we construct an s-cobordism between pairs $(M, L) \notin (S^{n+2}, K)$ and (M, L) and apply lemma 1.6. We denote the set of all null-cobordant *n*-knots by \mathscr{K}_n^0 . According to Kervaire [K] (cf. [KW, Chap. IV]) $\mathscr{K}_n = \mathscr{K}_n^0$ if *n* is even, but $\mathscr{K}_n \neq \mathscr{K}_n^0$ if *n* is odd.

PROPOSITION 3.6. Suppose $\langle m \rangle = 1$ for (M^{n+2}, L^n) and $n \ge 3$. Then $I_0(M, L)$ contains \mathscr{K}_n^0 . In particular, if n is even ≥ 4 , then $I_0(M, L) = I(M, L) = \mathscr{K}_n$.

Proof. Let (S^{n+2}, K) bound a disk pair (D^{n+3}, D) , where D is a (n+1)-disk. The boundary connected sum $(M, L) \times I \nmid (D^{n+3}, D)$ at the 1-level gives a cobordism between (M, L) and $(M, L) \notin (S^{n+2}, K)$.

We shall check the conditions (1) and (2) in Lemma 1.6 for this cobordism. First, since D is diffeomorphic to D^{n+1} , $L \times I \nmid D$ is diffeomorphic to $L \times I$; so (1) is satisfied. Hence $E(L \times I \nmid D)$ gives a cobordism relative boundary between E(L) and $E(L \not K)$. We note that

$$(3.7) E(L \times I \natural D) = E(L \times I) \cup E(D)$$

where $E(L \times I)$ and E(D) are pasted together along $D^{n+1} \times S^1$ embedded in their boundaries. The S^1 factor corresponds to meridians of $L \times I$ and D. Then the van Kampen's theorem says that

$$\pi_1(E(L \times I \nmid D)) \simeq \pi_1(E(L \times I)) \underset{}{*} \pi_1(E(D))$$
$$\simeq \pi_1(E(L \times I)) * (\pi_1(E(D))/)$$

where the latter isomorphism is because $\langle m \rangle = 1$ in $\pi_1(E(L \times I))$ by the assumption. Since $\pi_1(E(D))/\langle m \rangle \simeq \pi_1(D^{n+3}) \simeq \{1\}$, we have

(3.8)
$$\pi_1(E(L \times I \natural D)) \simeq \pi_1(E(L \times I)) \simeq \pi_1(E(L)) .$$

Here the inclusion map $i: E(L) = E(L) \times \{0\} \rightarrow E(L \times I \nmid D)$ induces the isomorphism.

We shall observe that *i* is a simple homotopy equivalence. For that purpose we consider the lifting of *i* to the universal covers. Since the map $\pi_1(E(D)) \rightarrow \pi_1(E(L \times I \nmid D))$ induced by the inclusion map is trivial as observed above, it follows from (3.7) that

(3.9)
$$\widetilde{E}(L \times I \nmid D) = \widetilde{E}(L \times I) \cup E(D) \times \Pi$$

where $\Pi = \pi_1(E(L \times I \nmid D)) = \pi_1(M-L)$ and $\tilde{E}(L \times I)$ and $E(D) \times \Pi$ are pasted together Π -equivariantly along $D^{n+1} \times S^1 \times \Pi$ embedded in their boundaries. This means that $\tilde{i}_*: H_q(\tilde{E}(L); \mathbb{Z}) \to H_q(\tilde{E}(L \times I \nmid D); \mathbb{Z})$ is an isomorphism as $\mathbb{Z}[\Pi]$ -modules. Hence $i_*: \pi_q(E(L)) \to \pi_q(E(L \times I \nmid D))$ is an isomorphism by Namioka's theorem (see [W11, §4]) and hence *i* is a homotopy equivalence.

The assumption $\langle m \rangle = 1$ together with (3.9) tells us that the Whitehead torsion $\tau(i) \in Wh(\Pi)$ of the map *i* comes from an element of Wh(1) through the map: $Wh(1) \to Wh(\Pi)$ induced from the inclusion $1 \to \Pi$. However Wh(1) = 0 and hence $\tau(i) = 0$. This shows that $E(L \times I \nmid D)$ is an s-cobordism relative boundary. The proposition then follows from Lemma 1.6. Q.E.D.

Proposition 3.6 gives a complete answer to the case where n is even ≥ 4 . It would be interesting to ask if the same conclusion still holds in the case n = 2.

In the next section we will improve Proposition 3.6 when n is odd ≥ 5 .

§ 4. AN IMPROVEMENT

Throughout this section we assume *n* is odd ≥ 5 . Let V^{n+1} be a Seifert surface of an *n*-knot *K* in S^{n+2} . The normal bundle to *V* in S^{n+2} is trivial. We give the stable normal bundle of S^{n+2} a canonical framing so that *V* can be viewed as a framed manifold.

Remember that $\partial V = K = S^n$. We make V contractible by framed surgery without touching the boundary. As is well known this is always possible in case dim V = n + 1 is odd. But in case n + 1 is even, we encounter an obstruction which is detected by

$$\begin{cases} \text{Sign } V \in \mathbb{Z} & \text{if } n+1 \equiv 0 \text{ (4)} \\ c(V) \in \mathbb{Z}/2\mathbb{Z} & \text{if } n+1 \equiv 2 \text{ (4)} \end{cases}$$

where c(V) is the Kervaire invariant of V.

Remark 4.1. Since ∂V is diffeomorphic to S^n , c(V) = 0 if n + 1 is not of the form $2^k - 2$ ([Br]).

One can see that Seifert surfaces of K are framed cobordant relative boundary to each other. Hence the values Sign V and c(V) are independent of the choice of V. We set

$$\sigma(S^{n+2}, K) = \begin{cases} \text{Sign } V & \text{if } n+1 \equiv 0 \ (4), \\ c(V) & \text{if } n+1 = 2^k - 2 \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 4.2. Suppose $\langle m \rangle = 1$ for (M^{n+2}, L^n) and n is odd ≥ 5 . Then $(S^{n+2}, K) \in I_0(M, L)$ if $\sigma(S^{n+2}, K) = 0$. In particular, $I_0(M, L) = \mathscr{K}_n$ if neither $n + 1 \equiv 0$ (4) nor $n + 1 = 2^k - 2$ for some k.

Combining this with Theorem 1.1, we obtain

COROLLARY 4.3. Suppose $\langle m \rangle = 1$ for (M^{n+2}, L^n) and $n+1 \equiv 0$ (4) $(n \neq 3)$. Then $(S^{n+2}, K) \in I_0(M, L)$ if and only if $\sigma(S^{n+2}, K) = 0$.

The rest of this section is devoted to the proof of Proposition 4.2. Let K be an *n*-knot in S^{n+2} such that $\sigma(S^{n+2}, K) = 0$. We shall construct an s-cobordism relative boundary between $E(L \ K)$ and E(L). The argument is rather more complicated than that of Proposition 3.6. We need some knowledge of surgery theory.

Step 1. Let V^{n+1} be a Seifert surface of K. Push the interior of V into the interior of D^{n+3} to make it transverse to the boundary S^{n+2} of D^{n+3} . We may assume that V is (n-1)/2-connected, if necessary, by doing framed surgery of V within D^{n+3} . In fact, this is the method used to prove that any n-knot is concordant to a simple knot (see [KW, Chap. IV]).

In the attempt to make V(n+1)/2-connected (and hence V is contractible by the Poincaré duality) by framed surgery of V within D^{n+3} , one encounters an obstruction. Namely a bunch of embedded (n+1)/2-spheres in V does not necessarily extend to embedded (n+3)/2-disks whose interior lies in $D^{n+3} - V$.

But if we do framed surgery of V at the outside of D^{n+3} without touching boundary, i.e. if we do surgery on framed embeddings

$$(S^{(n+1)/2} \times D^{(n+1)/2} \times D^2, S^{(n+1)/2} \times D^{(n+1)/2} \times \{0\}) \to (D^{n+3}, V),$$

then we can make V(n+1)/2-connected because the obstruction is exactly $\sigma(S^{n+2}, K)$ and it vanishes by the assumption. The ambient space is, however, not D^{n+3} any more. We denote by (W, D) the resulting framed oriented pair, where D is diffeomorphic to D^{n+1} .

Step 2. We construct a boundary preserving map h:

$$(W; N(D), E(D)) \rightarrow (D^{n+3}; N(D^{n+1}), E(D^{n+1}))$$

such that

(4.4)
$$h|_{\partial W}: \partial W = S^{n+2} \to \partial D^{n+3} = S^{n+2}$$
 is a homotopy equivalence,

(4.5)
$$h|_{N(D)}: N(D) \to N(D^{n+1})$$
 is a diffeomorphism,

where N denotes a closed tubular neighborhood and $D^{n+1} \subset D^{n+3}$ is standardly embedded.

Since D is diffeomorphic to D^{n+1} , there is a diffeomorphism

$$g: (D^{n+1} \times D^2, D^{n+1} \times \{0\}) \to (N(D), D).$$

Here $D^{n+1} \times D^2$ can be naturally identified with $N(D^{n+1})$; so we define (4.6) $h|_{N(D)} = g^{-1}$

First we extend $h|_{\partial W \cap \partial N(D)} = h|_{\partial E(K)}$ to a map from E(K) to $E(\partial D^{n+1}) = E(S^n)$. The obstruction lies in groups

$$H^{q+1}(E(K), \partial E(K); \pi_q(E(S^n)))$$
.

Since $E(S^n)$ is homotopy equivalent to S^1 , it suffices to prove

(4.7)
$$H^{q+1}(E(K), \partial E(K); \mathbf{Z}) = 0 \quad \text{for} \quad q = 0, 1.$$

On the other hand we have

$$H^{q+1}(E(K), \partial E(K); \mathbf{Z}) \simeq H^{q+1}(S^{n+2}, N(K); \mathbf{Z}) \quad \text{(by excision)}$$

$$\simeq \tilde{H}^{q}(N(K); \mathbf{Z}) \quad \text{(if } q+1 < n+2)$$

$$\simeq \tilde{H}^{q}(S^{n}; \mathbf{Z})$$

$$= 0 \quad \text{(if } q \neq n)$$

Hence (4.7) is satisfied as $n \ge 5$.

Consequently we can extend $h|_{N(D)}$ to a map

 $h|_{N(D)\cup\partial W}: (N(D)\cup\partial W, \partial W) \to (N(D^{n+1})\cup\partial D^{n+3}, \partial D^{n+3}).$

The local degree of $h|_{\partial W}: \partial W \to \partial D^{n+3}$ is one because $h|_{\partial W \cap N(D)} = h|_{N(K)}:$ $N(K) \to N(S^n)$ is a diffeomorphism by (4.6) and $h(E(K)) \subset E(S^n)$ by the construction. Since ∂W and ∂D^{n+3} are both S^{n+2} , $h|_{\partial W}$ is a homotopy equivalence. Hence (4.4) is satisfied. Moreover (4.5) is also satisfied by (4.6). In the sequel it suffices to extend $h|_{\partial E(D)}$ to a map from E(D) to $E(D^{n+1})$. This time the obstruction lies in groups

$$H^{q+1}(E(D), \partial E(D); \pi_q(E(D^{n+1})))$$

Since $E(D^{n+1})$ is homotopy equivalent to S^1 , it suffices to prove

(4.8)
$$H^{q+1}(E(D), \partial E(D); \mathbf{Z}) = 0 \quad \text{for} \quad q = 0, 1$$

By excision we have

$$H^{q+1}(E(D), \partial E(D); \mathbf{Z}) \simeq H^{q+1}(W, N(D) \cup \partial W; \mathbf{Z}).$$

Remember that W is obtained from D^{n+3} by (n+1)/2-surgery. It implies that

$$\tilde{H}^{i}(W; \mathbf{Z}) = 0$$
 if $i \neq (n+1)/2 + 1$.

In particular

$$\tilde{H}^i(W; \mathbf{Z}) = 0 \quad \text{for} \quad i \leq 3$$

as $n \ge 5$. Therefore it follows from the exact sequence of the pair $(W, N(D) \cup \partial W)$ that

$$H^{q+1}(W, N(D) \cup \partial W; \mathbf{Z}) \simeq \widetilde{H}^{q}(N(D) \cup \partial W; \mathbf{Z}) \quad \text{for} \quad q \leq 2.$$

Here the Mayer-Vietoris exact sequence of the triad $(N(D) \cup \partial W; N(D), \partial W)$ shows that

$$\widetilde{H}^{q}(N(D)\cup\partial W;\mathbf{Z}) = 0 \quad \text{for} \quad q = 0, 1,$$

because N(D) is contractible, $\partial W = S^{n+2}$, and $N(D) \cap \partial W = S^n \times S^1$. Hence (4.8) is satisfied, and we have obtained the desired map h.

Step 3. Since W is framed, the framing of the stable normal bundle v(W) of W induces a stable bundle map $b: v(W) \to v(D^{n+3})$ which covers h. The triple $\mathscr{B} = (W, h, b)$ is called a normal map.

The identity map $Id: (M, L) \times I \to (M, L) \times I$ gives a normal map where the stable bundle map is also the identity. We shall denote the normal map by $\mathscr{B}_{Id} = ((M, L) \times I, Id, Id)$. The maps h and Id are both diffeomorphisms on N(D) and $N(L \times I)$ respectively; so one can do the boundary connected sum of \mathscr{B} and \mathscr{B}_{Id} at points of K and $L \times \{1\}$. This yields a new normal map $\mathscr{B}_{Id} \nmid \mathscr{B} = (M \times I \nmid W, Id \nmid h, Id \restriction b)$. Here we naturally identify the target space $(M, L) \times I \nmid (D^{n+3}, D^{n+1})$ with $(M, L) \times I$. Since $Id \ddagger h$ is a diffeomorphism on $N(L \times I \nmid D)$, it gives a product structure on $N(L \times I \ddagger D)$. Thus we get a cobordism $E(L \times I \nmid D)$ relative boundary between $E(L \ddagger K)$ and E(L).

Step 4. $Id \nmid h|_{E(L)} \colon E(L) \to E(L) \times \{0\}$ (the 0-level) is the identity; so it is a simple homotopy equivalence. We shall observe that $h_1 = Id \nmid h|_{E(L \# K)} \colon E(L \# K) \to E(L) \times \{1\}$ (the 1-level) is also a simple homotopy equivalence.

We have a decomposition

$$E(L \ \# K) = E(L) \cup E(K)$$

in the same sense as (3.7). Hence, similarly to (3.8) one can see

(4.9)
$$\pi_1(E(L \ \sharp K)) \simeq \pi_1(E(L))$$

where the inclusion map induces the isomorphism.

We can view $E(L) \times \{1\}$ as $E(L \notin S^n)$ and we also have

$$E(L \# S^n) = E(L) \cup E(S^n).$$

Then the map h_1 can be viewed as the identity on E(L) and h on E(K). This together with (4.9) shows that $h_{1*}: \pi_1(E(L \# K)) \to \pi_1(E(L \# S^n))$ is an isomorphism.

As before we consider the map $\tilde{h_1}: \tilde{E}(L \# K) \to \tilde{E}(L \# S^n)$ lifted to the universal covers. Since $\langle m \rangle = 1$, we have a diagram

(4.10)
$$\widetilde{E}(L \, \sharp \, K) = \widetilde{E}(L) \cup E(K) \times \Pi$$
$$\overset{\tilde{h}_1}{\downarrow} \qquad \downarrow^{Id} \qquad \downarrow^{h|_{E(K)} \times Id}$$
$$\widetilde{E}(L^{\, \sharp} \, S^n) = \widetilde{E}(L) \cup E(S^n) \times \Pi ,$$

where $\Pi = \pi_1(M-L)$ as before. Since $h|_{E(K)}$ is a homology equivalence, the above diagram tells us that $\tilde{h}_{1*}: H_q(\tilde{E}(L \# K); \mathbb{Z}) \to H_q(\tilde{E}(L \# S^n); \mathbb{Z})$ is an isomorphism as $\mathbb{Z}[\Pi]$ -modules. Therefore h_1 is a homotopy equivalence by the same reason as before.

The assumption $\langle m \rangle = 1$ together with the above diagram tells us that $\tau(h_1) \in Wh(\Pi)$ comes from an element of Wh(1). Hence $\tau(h_1) = 0$ as Wh(1) = 0.

Step 5. By step 4 $\overline{h} = Id ||h||_{E(L \times I ||D|)}$: $E(L \times I ||D|) \to E(L \times I ||D|^{n+1})$ = $E(L \times I)$ is a simple homotopy equivalence on the boundary. We convert \overline{h} into a simple homotopy equivalence by surgery without touching the boundary. The obstruction $\sigma(\overline{h})$ lies in an L-group $L_{n+3}(\Pi, 1)$ where 1 denotes the trivial homomorphism from Π to \mathbb{Z}_2 (note, since M is oriented and hence so is $E(L \times I)$, the orientation homomorphism: $\Pi = \pi_1(E(L \times I)) \to \mathbb{Z}_2$ is trivial).

We have a diagram similar to (4.10):

$$E(L \times I \nmid D) = E(L \times I) \cup E(D)$$

$$\bar{h} \downarrow \qquad \qquad \downarrow^{Id} \qquad \downarrow^{h}$$

$$E(L \times I \nmid D^{n+1}) = E(L \times I) \cup E(D^{n+1}).$$

The surgery obstruction $\sigma(h)$ to converting h to a simple homotopy equivalence by surgery without touching the boundary lies in $L_{n+3}(\mathbb{Z}, 1)$ because $\pi_1(E(D^{n+1}))$ is isomorphic to \mathbb{Z} . The above diagram together with the assumption $\langle m \rangle = 1$ tells us that

$$\sigma(h) = \beta_* \alpha_* \sigma(h)$$

where $\alpha_*: L_{n+3}(\mathbb{Z}, 1) \to L_{n+3}(1, 1)$ and $\beta_*: L_{n+3}(1, 1) \to L_{n+3}(\Pi, 1)$ are the homomorphisms induced from the trivial homomorphisms $\alpha: \mathbb{Z} \to 1$ and $\beta: 1 \to \Pi$ respectively. It is well-known that

$$L_{n+3}(1, 1) \simeq \begin{cases} \mathbf{Z} & \text{if } n+3 \equiv 0 \ (4), \\ \mathbf{Z}_2 & \text{if } n+3 \equiv 2 \ (4). \end{cases}$$

As easily observed $\alpha_*\sigma(h)$ is given by

$$\begin{cases} \text{Sign } W & \text{if } n+3 \equiv 0 \ (4) \\ c(W) & \text{if } n+3 \equiv 2 \ (4) \end{cases}$$

through the above isomorphism. Remember that W is framed cobordant to D^{n+3} relative boundary by the construction. Therefore those invariants vanish and hence $\sigma(\bar{h}) = 0$.

Consequently we have obtained a cobordism U' relative boundary between E(L # K) and E(L) together with a simple homotopy equivalence $F: U' \to E(L \times I)$ which is the identity on the 0-level. Let $i_0: E(L) \to U'$ and $j_0: E(L) \to E(L \times I)$ be the inclusion maps from the 0-level to the cobordisms. Since $F \circ i_0 = j_0 \circ Id$ where $Id: E(L) \to E(L)$ denotes the identity map, we have

$$\tau(F) + F_*\tau(i_0) = \tau(j_0) + j_{0*}\tau(Id)$$

(see [M1, Lemma 7.8]). Here F, j_0 , and Id are all simple homotopy equivalences; so these Whitehead torsions vanish. Hence it follows that $\tau(i_0) = 0$, because $F_*: Wh(\pi_1(U')) \to Wh(\pi_1(E(L \times I)))$ is an isomorphism. This means that U' is an s-cobordism. Therefore $(S^{n+2}, K) \in I_0(M, L)$ by Lemma 1.6. Q.E.D.

§ 5. Type 3 case

In this section we treat the case where $\langle m \rangle$ or [m] is of order p (p is not necessarily a prime number). We begin with

LEMMA 5.1. Suppose [m] is of order p. Then if $(S^{n+2}, K) \in I(M, L)$, then $(S^{n+2}, K)_p$ is a homotopy (n+2)-sphere.

Proof. Let r be the order of Tor $H_1(M-L; \mathbb{Z})$, and let γ be the canonical epimorphism $\pi_1(M-L) \to H_1(M-L; \mathbb{Z}) \otimes \mathbb{Z}_r$. Since the order of $\gamma(\langle m \rangle)$ is p, we obtain the desired result by an argument similar to the proof of Lemma 2.1. Q.E.D.

If $p \ge 2$, there are infinitely many knots (S^{n+2}, K) such that $(S^{n+2}, K)_p$ is not a homotopy (n+2)-sphere; so Lemma 5.1 shows that $I(M, L) \subset \mathcal{K}_n$ for such (M, L).

The rest of this section is devoted to looking for a non-trivial knot in I(M, L) or $I_0(M, L)$. We will extend Proposition 3.6 and 4.2 to the case where $\langle m \rangle$ is of order p. Lemma 5.1 reminds us of counterexamples to the generalized Smith conjecture.

Let (S^{n+2}, K) be an *n*-knot which bounds a disk pair (D^{n+3}, D) such that $(D^{n+3}, D)_p$ is a homotopy (n+3)-disk. Since $(S^{n+2}, K)_p$ is the boundary of $(D^{n+3}, D)_p$, $(S^{n+2}, K)_p$ is a homotopy (n+2)-sphere. If $n + 3 \ge 5$, then $(D^{n+3}, D)_p$ is diffeomorphic to D^{n+3} and hence $(S^{n+2}, K)_p$ is diffeomorphic to S^{n+2} .

The p-fold branched cyclic covering $(D^{n+3}, D)_p$ supports a \mathbb{Z}_p -action with the branch set D as the fixed point set. Let $E(D)_p$ be the exterior of Din $(D^{n+3}, D)_p$ and let $\rho: S^1 \to E(D)_p$ be an equivariant embedding of a meridian of D in $E(D)_p$, where the standard free \mathbb{Z}_p -action is considered on S^1 . Since ρ is a homology equivalence and equivariant, the Whitehead torsion of ρ is defined in $Wh(\mathbb{Z}_p)$. Clearly it is independent of the choice of ρ ; so we shall denote it by $\tau_p(D^{n+3}, D)$.

The following theorem is an extension of Proposition 3.6.

THEOREM 5.2. Suppose $\langle m \rangle$ is of order p (p may be equal to 1) for (M^{n+2}, L^n) and $n \ge 4$. Then $(S^{n+2}, K) \in I_0(M, L)$ if it bounds a disk pair (D^{n+3}, D) such that

- (1) $(D^{n+3}, D)_p$ is diffeomorphic to D^{n+3} ,
- (2) $\mu_*\tau_p(D^{n+3}, D) = 0,$

where $\mu_*: Wh(\mathbb{Z}_p) \to Wh(\pi_1(M-L))$ is the homomorphism induced from a homomorphism $\mu: \mathbb{Z}_p \to \pi_1(M-L)$ sending a generator of \mathbb{Z}_p to $<m> \in \pi_1(M-L)$.

Remark 5.3. (1) For each p, there are infinitely many *n*-knots satisfying the conditions (1) and (2) in Theorem 5.2. For example the \mathbb{Z}_p -orbit spaces of Sumners' knots [R, p. 347] (which are counterexamples to the generalized Smith conjecture) are the desired knots. In fact, $\tau_p(D^{n+3}, D) = 0$ for them.

(2) If p = 1, 2, 3, 4, or 6, then $Wh(\mathbb{Z}_p) = 0$. Hence the condition (2) of Theorem 5.2 is trivially satisfied in these cases.

Proof of Theorem 5.2. We shall observe that the proof of Proposition 3.6 works with a little modification. As before $E(L \times I \nmid D)$ can be viewed as a cobordism relative boundary between E(L) and $E(L \not\equiv K)$. We shall check that this is an s-cobordism.

The condition (1) implies that

(5.4)
$$\pi_1(E(D))/\langle m^p \rangle \simeq \mathbb{Z}_p$$

where a meridian of D in D^{n+3} is also denoted by m. Hence it follows from the decomposition (3.7) that

(5.5)
$$\pi_1(E(L \times I \nmid D)) \simeq \pi_1(E(L \times I)) \underset{}{*} \pi_1(E(D))$$
$$\simeq \pi_1(E(L \times I)) \underset{\mathbf{Z}_p}{*} \pi_1(E(D)) / < m^p >$$
$$(as is of order p in \pi_1(E(L \times I)))$$
$$\simeq \pi_1(E(L \times I)) \qquad (by (5.4))$$

This implies that the inclusion map $i: E(L) = E(L) \times \{0\} \to E(L \times I \nmid D)$ induces an isomorphism $\pi_1(E(L)) \to \pi_1(E(L \times I \nmid D))$.

We consider the map $\tilde{i}: \tilde{E}(L) \to \tilde{E}(L \times I \nmid D)$ lifted to the universal cover. Let $q: \tilde{E}(L \times I \nmid D) \to E(L \times I \nmid D)$ be the covering projection map. By (5.5) $q^{-1}(E(L \times I))$ is exactly the universal cover $\tilde{E}(L \times I)$. As for $q^{-1}(E(D))$ we need a little consideration. The above observation (5.5) shows that the image of $j_*: \pi_1(E(D)) \to \pi_1(E(L \times I \nmid D))$ is isomorphic to \mathbb{Z}_p , where j is the inclusion map. We shall identify $j_*\pi_1(E(D))$ with \mathbb{Z}_p . Remember that \mathbb{Z}_p acts freely on $E(D)_p$ as covering transformations.

Claim 5.6. $q^{-1}(E(D)) = E(D)_p \underset{\mathbf{Z}_p}{\times} \Pi$, where the right hand side denotes the orbit space of $E(D)_p \times \Pi$ by the diagonal \mathbf{Z}_p -action defined by $s \cdot (x, g) = (xs^{-1}, sg)$ for $s \in \mathbf{Z}_p$, $x \in E(D)_p$, and $g \in \Pi$.

Proof. The Π -covering $q^{-1}(E(D)) \to E(D)$ is classified by the map: $E(D) \to B\Pi$ induced from the homomorphism $j_*: \pi_1(E(D)) \to \Pi = \pi_1(E(L \times I \nmid D))$. Here j_* factors through the inclusion $\& : \mathbb{Z}_p \to \Pi$:

$$\pi_1(E(D)) \xrightarrow{j_*} \Pi$$

$$\ell \bigvee \int_{U} \int_{U} \ell$$

$$\mathbf{Z}_p$$

The pullback of the universal Π -bundle $E\Pi \to B\Pi$ by \mathscr{k} is of the form $E\mathbb{Z}_p \underset{\mathbb{Z}_p}{\times} \Pi \to B\mathbb{Z}_p$. In fact, since $E\mathbb{Z}_p = E\Pi$, the map $(u, g) \to ug$ $(u \in E\mathbb{Z}_p, g \in \Pi)$ is defined from $E\mathbb{Z}_p \underset{\mathbb{Z}_p}{\times} \Pi$ to $E\Pi$. The map induces a Π -bundle map from $E\mathbb{Z}_p \underset{\mathbb{Z}_p}{\times} \Pi \to B\Pi$ to $E\Pi \to B\Pi$. On the other hand the covering induced from the homomorphism $\mathscr{l} : \pi_1(E(D)) \to \mathbb{Z}_p$ is exactly the \mathbb{Z}_p -covering $E(D)_p \to E(D)$. These prove the claim.

Consequently we have a decomposition

(5.7)
$$\widetilde{E}(L \times I \models D) = \widetilde{E}(L \times I) \cup E(D)_p \underset{\mathbf{Z}_p}{\times} \Pi,$$

where $\tilde{E}(L \times I)$ and $E(D)_p \underset{\mathbf{Z}_p}{\times} \Pi$ are pasted together along $D^n \times S^1 \underset{\mathbf{Z}_p}{\times} \Pi$ equivariantly embedded in their boundaries. The condition (1) means that $E(D)_p$ is a homology circle. This together with (5.7) tells us that $\tilde{i}: \tilde{E}(L \times I)$ $\rightarrow \tilde{E}(L \times I \nmid D)$ induces an isomorphism on homology as $\mathbf{Z}[\Pi]$ -modules. Hence *i* is a homotopy equivalence.

The decomposition (5.7) also tells us that

$$\tau(i) = \mu_* \tau_p(D^{n+3}, D) \quad \text{up to sign}.$$

Hence $\tau(i) = 0$ by the condition (2). Therefore $E(L \times I \nmid D)$ is an s-cobordism relative boundary. The theorem then follows from Lemma 1.6. Q.E.D.

A torsion $\tau_p(S^{n+2}, K)$ is defined similarly to $\tau_p(D^{n+3}, D)$ if $(S^{n+2}, K)_p$ is a homotopy (n+2)-sphere. The following theorem is an extension of Proposition 4.2.

THEOREM 5.8. Suppose $\langle m \rangle$ is of order p (p may be equal to 1) for (M^{n+2}, L^n) and $n \ge 4$. Let $a_{n,p} = 2$ if $n \equiv 0$ (4) and p is even, and let $a_{n,p} = 1$ otherwise. Then $a_{n,p}(S^{n+2}, K) \in I_0(M, L)$ if

(1) $\sigma(S^{n+2}, K) = 0$ in case *n* is odd.

(2) $(S^{n+2}, K)_p$ is a homotopy (n+2)-sphere,

(3)
$$a_{n, p} \mu_* \tau_p(S^{n+2}, K) = 0$$

where μ_* is the same as in Theorem 5.2.

Proof. The argument developed in Steps 1, 2, and 3 of the proof of Proposition 4.2 still works. Step 4 needs a little modification. Instead of (4.10) we have

(5.9)

$$\widetilde{E}(L \ \# K) = \widetilde{E}(L) \cup E(K)_{p} \underset{\mathbf{Z}_{p}}{\times} \Pi$$

$$\widetilde{h}_{1} \downarrow \qquad \downarrow^{Id} \qquad \downarrow^{h_{p} \times Id}$$

$$\widetilde{E}(L \ \# S^{n}) = \widetilde{E}(L) \cup E(S^{n})_{p} \underset{\mathbf{Z}_{p}}{\times} \Pi$$

(see (5.7)) where $h_p: E(K)_p \to E(S^n)_p$ denotes the lifting of h to the \mathbb{Z}_p -covers. Since h_p is a homology equivalence, the above diagram tells us that \tilde{h}_1 is a homotopy equivalence.

It also tells us that

$$\tau(h_1) = - \mu_* \tau_p(S^{n+2}, K) ,$$

which vanishes by the condition (3). Hence $h_1: E(L \# K) \to E(L \# S^n)$ is a simple homotopy equivalence.

Step 5 also needs some modification. We need to replace α and β by the canonical epimorphism $\gamma: \mathbb{Z} \to \mathbb{Z}_p$ and $\mu: \mathbb{Z}_p \to \Pi$ respectively. Then we have

$$\sigma(\overline{h}) = \mu_* \gamma_* \sigma(h) \, .$$

Here we distinguish three cases to observe the value $\sigma(h)$.

Case 1. The case where *n* is odd. In this case the trivial homomorphism $\alpha: \mathbb{Z} \to 1$ induces an isomorphism $L_{n+3}(\mathbb{Z}, 1) \to L_{n+3}(1, 1)$ ([Wl1, 13A.8]). As observed in Step 5 of the proof of Proposition 4.2, $\alpha_*(\sigma(h))$ vanishes. Hence $\sigma(h) = 0$, so $\sigma(\bar{h}) = 0$.

Case 2. The case where $n \equiv 2$ (4) or p is odd. According to Wall [W12] or Bak [Ba], $L_{n+3}(\mathbb{Z}_p, 1) = 0$ in this case. Since $\gamma_*\sigma(h)$ lies in $L_{n+3}(\mathbb{Z}_p, 1)$, $\gamma_*\sigma(h) = 0$ and hence $\sigma(\bar{h}) = 0$.

Case 3. The case where $n \equiv 0$ (4) and p is even. In this case $L_{n+3}(\mathbb{Z}_p, 1) \simeq \mathbb{Z}_2$. Since the value $\gamma_* \sigma(h) \in L_{n+3}(\mathbb{Z}_p, 1)$ is additive with respect to connected sum, it necessarily vanishes for $(S^{n+2}, K) \notin (S^{n+2}, K)$.

The rest of the argument is the same as that in Step 5. This proves the theorem. Q.E.D.

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