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# HURWITZ-RADON MATRICES AND PERIODICITY MODULO 8

## by Beno ECKMANN

### 0. Introduction

0.1. We consider complex  $n \times n$  — matrices  $A_1, A_2, ..., A_s$ , either all unitary (case U) or all orthogonal (case O); they are called Hurwitz-Radon matrices, in short HR-matrices, if

(1) 
$$A_j^2 = -E, A_jA_k + A_kA_j = 0, \quad j, k = 1, 2, ..., s, j \neq k;$$

E or  $E_n$  denotes the unit matrix. Such matrices are well-known to exist, even with entries  $0, \pm 1, \pm i$  (case U) or  $0, \pm 1$  (case O). The possible values of n are multiples  $mn_0$ , m = 1, 2, 3, ... where in case  $U, n_0 = 2^{s/2}$  if s is even,  $n_0 = 2^{(s-1)/2}$  if s is odd. In case  $O, n_0 = 2^{(s-1)/2}$  if  $s \equiv 7 \mod 8$ ;  $n_0 = 2^{s/2}$  if  $s \equiv 0, 6$ ;  $n_0 = 2^{(s+1)/2}$  if  $s \equiv 1, 3, 5$ ; and  $n_0 = 2^{(s+2)/2}$  if  $s \equiv 2, 4 \mod 8$ .

If we put  $A_0 = E$  the relations (1) are equivalent to

$$f_s(x_0, x_1, ..., x_s) = \sum_{j=0}^{s} x_j A_j$$

being a unitary, or orthogonal respectively, matrix for all real  $x_j$  with  $\sum_{0}^{s} x_j^2 = 1$ . Thus  $f_s$  can be considered as a map  $S^s \to U$  via U(n), or  $S^s \to O$  via O(n) where  $U = \lim_{s \to 0} U(k)$  and  $O = \lim_{s \to 0} O(k)$  are the infinite unitary and orthogonal groups. We also write  $f_s$  for the homotopy class of that map,  $f_s \in \pi_s(U)$  or  $\pi_s(O)$ . We recall that by the Bott periodicity theorems these groups are cyclic or 0.

Theorem A. If  $A_1$ ,  $A_2$ , ...,  $A_s$  are HR-matrices of minimal size  $n=n_0(s)$  then  $f_s$  is a generator of  $\pi_s(U)$ , or  $\pi_s(O)$  respectively, s=0,1,2,....

Remark 1. For s=0 (empty set of HR-matrices) we have  $f_0(x_0)=x_0(1)$ ,  $x_0^2=1$ ; i.e.,  $f_0(1)=(1)$ ,  $f_0(-1)=(-1)$ ,  $f_0:S^0\to O(1)\to O$ . For s>0,  $f_0:S^s\to O$  clearly factors through  $SO(n)\to SO$  (U being connected, the analogue is irrelevant in the unitary case).

Remark 2. The problem originally solved by Hurwitz [H] and Radon [R] refers to the case O: One asks for complex bilinear forms z = f(x, y) =  $(\sum_{0}^{s} x_{j}A_{j})y$ , where  $z = (z_{1}, ..., z_{n}), y = (y_{1}, ..., y_{n}), x = (x_{0}, ..., x_{s})$ , such that  $z_{1}^{2} + ... + z_{n}^{2} = (x_{0}^{2} + ... + x_{s}^{2})(y_{1}^{2} + ... + y_{n}^{2})$ .

This means that  $\sum_{j=0}^{s} x_{j}A_{j}$  is orthogonal, i.e. leaves invariant  $\sum_{j=0}^{n} y_{j}^{2}$  except for the factor  $\sum_{j=0}^{s} x_{j}^{2}$ ; and thus, since we may assume  $A_{0} = E$ , that  $A_{1}, ..., A_{s}$  is a set of orthogonal HR-matrices in the sense of (1).

The case U refers to the analogous problem for the identity

$$|z_1|^2 + ... + |z_n|^2 = (x_0^2 + ... + x_s^2)(|y_1|^2 + ... + |y_n|^2)$$

where y and z are complex, and x real.

0.2. The symplectic case: It is also of interest to consider HR-matrices, i.e. matrices fulfilling (1), which are symplectic. A linear combination  $\sum_{j=0}^{s} x_{j}A_{j}$  of  $2n \times 2n$ -matrices with  $A_{0} = E$  is symplectic up to the factor  $\sum_{j=0}^{s} x_{j}^{2}$  if and only if  $A_{1}, ..., A_{s}$  is a set of symplectic HR-matrices (Proposition 4.1).

We restrict attention to unitary symplectic matrices, i.e., to the group  $Sp(n) \subset U(2n)$ , and write Sp for the infinite symplectic group  $\lim_{s \to \infty} Sp(k)$ . With a set  $A_1, ..., A_s$  of unitary symplectic HR-matrices, and  $A_0 = E$ , we associate the map  $f_s(x_0, x_1, ..., x_s) = \sum_{s=0}^{s} x_j A_j$ ,  $\sum_{s=0}^{s} x_j^2 = 1$ , of  $S^s$  into Sp via Sp(n); we also write  $f_s$  for the corresponding element of  $\pi_s(Sp)$ , known to be 0 or cyclic.

Theorem A'. If  $A_1$ , ...,  $A_s$  are unitary symplectic HR-matrices of minimal size  $2n_0$  then  $f_s$  is a generator of  $\pi_s(Sp)$ .

0.3. The paper is organized as follows. We first recall (Section 1) that the HR-matrix problem can be formulated in terms of representations of certain finite group  $G_s$ , s=0,1,2,... introduced by the author [E], and discuss these representations using the elegant description of [LS]. In Section 2 the "reduced" Grothendieck groups of representations  $E_s^U$  and  $E_s^O$  are

computed; they turn out to be isomorphic to  $\pi_s(U)$  and  $\pi_s(O)$  respectively. Moreover a product is defined in the direct sum of the  $E_s^U(E_s^O)$  turning it into a graded ring  $E_*^U(E_*^O)$ . The claim of Theorem A is proved in Section 3; we show that the maps  $\phi: E_s^U \to \pi_s(U)$ ,  $\psi: E_s^O \to \pi_s(O)$  given by the  $f_s$  of 0.1 are isomorphisms. Using the product structure in  $\pi_*(U)$  and  $\pi_*(O)$  known from K-theory the proof reduces to simple verifications in low dimensions. The symplectic case is dealt with in Section 4. In Section 5 we make a remark concerning the "linearization phenomenon" for the homotopy groups of U, O and Sp.

## 1. The groups $G_s$ and their representations

- We will denote throughout by  $G_s$  the group given by the presentation  $G_s = \langle \varepsilon, a_1, ..., a_s | \varepsilon^2 = 1, a_j^2 = \varepsilon, a_j a_k = \varepsilon a_k a_j, j, k = 1, 2, ..., s, j \neq k \rangle$ . Clearly any set  $A_1, ..., A_s$  of HR-matrices yields a (unitary or orthogonal) representation of  $G_s$  of degree n by  $\varepsilon \mapsto -E$ ,  $a_j \mapsto A_j$ , j = 1, 2, ..., s. Conversely a representation of  $G_s$  with  $\varepsilon \mapsto -E$ , in short an  $\varepsilon$ -representation, yields a set of s HR-matrices. For the elementary properties of  $G_s$  and its representations we refer to [E]. We just recall that the order of  $G_s$ is  $2^{s+1}$ , that  $\varepsilon$  is central, and that the irreducible unitary  $\varepsilon$ -representations of  $G_s$  are of degree  $2^{s/2}$  if s is even (one equivalence class), of degree  $2^{(s-1)/2}$  if s is odd (two equivalence classes). These degrees are the minimal values  $n_0$  in case U. As for the case O, one has to recall that a representation is equivalent to an orthogonal one if and only if it is equivalent to a real (and orthogonal) one. Thus, unless an irreducible unitary e-representation is already real, one has to add its conjugate-complex representation, and the discussion of the various cases depending on s yields the minimal values  $n_0$  (case O) mentioned in the introduction; in other words, the degrees
- 1.2. A very simple and useful scheme for studying the groups  $G_s$  and their  $\varepsilon$ -representations (and many other things) has been deviced by T. Y. Lam and T. Smith [LS]. They have expressed the  $G_s$  as products of very small and well-known groups. Namely  $C = G_1$ , the cyclic group of order 4;  $Q = G_2$ , the quaternionic group of order 8; K, the Klein 4-group; and D, the dihedral group of order 8. Although K and D do not belong to the family  $G_s$ , they are of a similar nature and contain a distinguished central element  $\varepsilon$  of order 2 (distinguished arbitrarily in K). "Product" here means the central product obtained from the direct product by identifying the

of the irreducible orthogonal  $\varepsilon$ -representations of  $G_{\varepsilon}$ .