1. The groups G_s and their representations

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computed; they turn out to be isomorphic to $\pi_s(U)$ and $\pi_s(O)$ respectively. Moreover a product is defined in the direct sum of the $E_s^U(E_s^O)$ turning it into a graded ring $E_*^U(E_*^O)$. The claim of Theorem A is proved in Section 3; we show that the maps $\phi: E_s^U \to \pi_s(U)$, $\psi: E_s^O \to \pi_s(O)$ given by the f_s of 0.1 are isomorphisms. Using the product structure in $\pi_*(U)$ and $\pi_*(O)$ known from K-theory the proof reduces to simple verifications in low dimensions. The symplectic case is dealt with in Section 4. In Section 5 we make a remark concerning the "linearization phenomenon" for the homotopy groups of U, O and Sp.

1. The groups G_s and their representations

1.1. We will denote throughout by G_s the group given by the presentation

$$G_s = <\varepsilon, a_1, ..., a_s \mid \varepsilon^2 = 1,\ a_j^2 = \varepsilon,\ a_j a_k = \varepsilon a_k a_j,\ j, k = 1, 2, ..., s, j \neq k> .$$

Clearly any set $A_1, ..., A_s$ of HR-matrices yields a (unitary or orthogonal) representation of G_s of degree n by $\varepsilon \mapsto -E$, $a_j \mapsto A_j$, $j = 1, 2, ..., s$. Conversely a representation of G_s with $\varepsilon \mapsto -E$, in short an ε-representation, yields a set of s HR-matrices. For the elementary properties of G_s and its representations we refer to [E]. We just recall that the order of G_s is 2^{s+1}, that ε is central, and that the irreducible unitary ε-representations of G_s are of degree $2^{s/2}$ if s is even (one equivalence class), of degree $2^{(s-1)/2}$ if s is odd (two equivalence classes). These degrees are the minimal values n_0 in case U. As for the case O, one has to recall that a representation is equivalent to an orthogonal one if and only if it is equivalent to a real (and orthogonal) one. Thus, unless an irreducible unitary ε-representation is already real, one has to add its conjugate-complex representation, and the discussion of the various cases depending on s yields the minimal values n_0 (case O) mentioned in the introduction; in other words, the degrees of the irreducible orthogonal ε-representations of G_s.

1.2. A very simple and useful scheme for studying the groups G_s and their ε-representations (and many other things) has been deviced by T. Y. Lam and T. Smith [LS]. They have expressed the G_s as products of very small and well-known groups. Namely $C = G_1$, the cyclic group of order 4; $Q = G_2$, the quaternionic group of order 8; K, the Klein 4-group; and D, the dihedral group of order 8. Although K and D do not belong to the family G_s, they are of a similar nature and contain a distinguished central element ε of order 2 (distinguished arbitrarily in K). "Product" here means the central product obtained from the direct product by identifying the two ε's. The expression for the G_s then is as follows, displaying a fundamental periodicity modulo 8:

(2)

s	0	1	2	3	4	5	6	7	8	9	...
G_s	$\mathbf{Z}/2$	C	Q	QK	QD	D^2C	D^3	D^3K	D^4	D^4C	...

and $G_{s+8} = D^4G_s$.

The tensor product of ε-representations of two of the groups G_s, K, D is an ε-representation of their product above, and all ε-representations of the G_s can be obtained in that explicit way from those of C, Q, K, D, which are well-known. This yields, in particular, the characters χ and the Schur indices I of the irreducible unitary ε-representation (the Schur index $I = 1$ if the representation is equivalent to a real one; if it is not, $I = -1$ if it is equivalent to the conjugate-complex one, $I = 0$ otherwise). Both χ and I behave multiplicatively with respect to the central product.

1.3. The Schur indices of the irreducible ε-representations are: 0 for $C = G_1$, -1 for $Q = G_2$, and 1 for K and D (two equivalence classes for K, one for D). This yields the Schur indices I_s of the irreducible ε-representations of the G_s, as listed in (2) below; we further list the numbers ν_s^U of inequivalent unitary, and ν_s^O of inequivalent orthogonal irreducible ε-representations, and the respective degrees d_s^U, d_s^O. Note that I_s is periodic with period 8, and d_s^O is periodic with period 8 in the sense that $d_{s+8}^O = 16d_s^O$. Finally we include in the same table the Grothendieck groups D_s^U and D_s^O of (equivalence classes of) irreducible ε-representations of G_s, with respect to the direct sum of representations.

(3)

s	0	1	2	3	4	5	6	7	8	9	...
I_s	1	0	−1	−1	−1	0	1	1	1	0	...
ν_s^U	1	2	1	2	1	2	1	2	1	2	
ν_s^O	1	1	1	2	1	1	1	2	1	1	
d_s^U	1	1	2	2	4	4	8	8	16	16	
d_s^O	1	2	4	4	8	8	8	8	16	32	
D_s^U	$\mathbf{Z}$	$\mathbf{Z}\oplus\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}\oplus\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}\oplus\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}\oplus\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}\oplus\mathbf{Z}$	
D_s^O	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}\oplus\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}\oplus\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	

The values of d_s^O follow immediately from the I_s and the d_s^U. The values n_0 for the case O, as given in the Introduction, are the d_s^O.

2. The reduced ε-representation ring

2.1. For all $s \geqslant 0$ the group G_s is the subgroup of G_{s+1} obtained by omitting the generator a_{s+1}; let $h_s \colon G_s \to G_{s+1}$ be the embedding homomorphism. Via h_s we can restrict an ε-representation of G_{s+1} to G_s, which in terms of HR-matrices means omitting A_{s+1}.

Let $h_s^* \colon D_{s+1}^U \to D_s^U$ be the corresponding homomorphism of Grothendieck groups, and $E_s^U = D_s^U / h_s^* D_{s+1}^U$ the "reduced" groups; similarly $E_s^O = D_s^O / h_s^* D_{s+1}^O$. They can easily be computed by means of the characters of ε-representations, as follows.

For Q and D the character of an irreducible unitary ε-representation is 0 except on 1 and ε. For C and K it is $\neq 0$ on all 4 elements; on the essential generator ($\neq$ε) of C it is $+i$ or $-i$ for the two inequivalent representations, and $+1$ or -1 in the case of K. For G_s, s even, we infer from the table (2) that the character is 0 except on 1, ε. For G_s, s odd, the character is 0 except on 1, ε and two further elements z, εz; on these the two inequivalent ε-representations differ just by the sign of the character.

If s is even, $d_{s+1}^U = d_s^U = 2^{s/2}$; thus the restriction of an irreducible ε-representation must be irreducible, whence $h_s^* D_{s+1}^U = D_s^U$, $E_s^U = 0$. If s is odd, $d_{s+1}^U = 2d_s^U = 2^{(s+1)/2}$; thus the restriction is the sum of two irreducible ε-representations, and since the character is 0 (except on 1, ε) these two must be inequivalent. Therefore $h_s^* D_{s+1}^U$ is the "diagonal" of $D_s^U = \mathbf{Z} \oplus \mathbf{Z}$, and $E_s^U = \mathbf{Z}$; its generator ρ_s is represented by either of the two inequivalent irreducible ε-representations of G_s, $-\rho_s$ by the other one.

In the orthogonal case the E_s^O are computed similarly from (3). Since $d_1^O = 2$ and $d_0^O = 1$, the restriction from D_1^O to D_0^O yields twice the generator, and $E_0^O = \mathbf{Z}/2$; the same argument holds for $s \equiv 0 \bmod 8$, $d_{s+1}^O = 2d_s^O$. Since $d_2^O = 4$ and $d_1^O = 2$, we get $E_1^O = \mathbf{Z}/2$. From $d_3^O = d_2^O = 4$ we get $E_2^O = 0$. As for $s = 3$, the character argument shows that $h_3^* D_4^O =$ diagonal of D_3^O $(= \mathbf{Z} \oplus \mathbf{Z})$, and $E_3^O = \mathbf{Z}$. For $s = 4, 5, 6$ the dimensions $d_{s+1}^O = d_s^O$ show that $E_4^O = E_5^O = E_6^O = 0$. For $s = 7$, the character argument yields $h_7^* D_8^O =$ diagonal of D_7^O $(= \mathbf{Z} \oplus \mathbf{Z})$, and $E_7^O = \mathbf{Z}$. Finally one has, for all s, $E_{s+8}^O \cong E_s^O$.