2. The reduced -representation ring

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **35 (1989)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **05.06.2024**

The values of d_s^O follow immediately from the I_s and the d_s^U. The values n_0 for the case O, as given in the Introduction, are the d_s^O.

2. The reduced ε-representation ring

2.1. For all $s \geqslant 0$ the group G_s is the subgroup of G_{s+1} obtained by omitting the generator a_{s+1}; let $h_s \colon G_s \to G_{s+1}$ be the embedding homomorphism. Via h_s we can restrict an ε-representation of G_{s+1} to G_s, which in terms of HR-matrices means omitting A_{s+1}.

Let $h_s^* \colon D_{s+1}^U \to D_s^U$ be the corresponding homomorphism of Grothendieck groups, and $E_s^U = D_s^U / h_s^* D_{s+1}^U$ the "reduced" groups; similarly $E_s^O = D_s^O / h_s^* D_{s+1}^O$. They can easily be computed by means of the characters of ε-representations, as follows.

For Q and D the character of an irreducible unitary ε-representation is 0 except on 1 and ε. For C and K it is $\neq 0$ on all 4 elements; on the essential generator ($\neq \varepsilon$) of C it is $+i$ or $-i$ for the two inequivalent representations, and $+1$ or -1 in the case of K. For G_s, s even, we infer from the table (2) that the character is 0 except on $1, \varepsilon$. For G_s, s odd, the character is 0 except on $1, \varepsilon$ and two further elements $z, \varepsilon z$; on these the two inequivalent ε-representations differ just by the sign of the character.

If s is even, $d_{s+1}^U = d_s^U = 2^{s/2}$; thus the restriction of an irreducible ε-representation must be irreducible, whence $h_s^* D_{s+1}^U = D_s^U$, $E_s^U = 0$. If s is odd, $d_{s+1}^U = 2d_s^U = 2^{(s+1)/2}$; thus the restriction is the sum of two irreducible ε-representations, and since the character is 0 (except on $1, \varepsilon$) these two must be inequivalent. Therefore $h_s^* D_{s+1}^U$ is the "diagonal" of $D_s^U = \mathbf{Z} \oplus \mathbf{Z}$, and $E_s^U = \mathbf{Z}$; its generator ρ_s is represented by either of the two inequivalent irreducible ε-representations of G_s, $-\rho_s$ by the other one.

In the orthogonal case the E_s^O are computed similarly from (3). Since $d_1^O = 2$ and $d_0^O = 1$, the restriction from D_1^O to D_0^O yields twice the generator, and $E_0^O = \mathbf{Z}/2$; the same argument holds for $s \equiv 0 \bmod 8$, $d_{s+1}^O = 2d_s^O$. Since $d_2^O = 4$ and $d_1^O = 2$, we get $E_1^O = \mathbf{Z}/2$. From $d_3^O = d_2^O = 4$ we get $E_2^O = 0$. As for $s = 3$, the character argument shows that $h_3^* D_4^O =$ diagonal of $D_3^O (= \mathbf{Z} \oplus \mathbf{Z})$, and $E_3^O = \mathbf{Z}$. For $s = 4, 5, 6$ the dimensions $d_{s+1}^O = d_s^O$ show that $E_4^O = E_5^O = E_6^O = 0$. For $s = 7$, the character argument yields $h_7^* D_8^O =$ diagonal of $D_7^O (= \mathbf{Z} \oplus \mathbf{Z})$, and $E_7^O = \mathbf{Z}$. Finally one has, for all s, $E_{s+8}^O \cong E_s^O$.

These results are summarized in the table

(4)

s	0	1	2	3	4	5	6	7	8	9	...
E_s^U	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	
E_s^O	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0	$\mathbf{Z}$	0	0	0	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	

According to the Bott periodicity theorems the above table is just that of the $\pi_s(U)$ and $\pi_s(O)$, $s = 0, 1, 2, \ldots$. Before studying the relation as stated in Theorem A we establish product structures in the reduced Grothendieck groups of ε-representations, i.e., of HR-matrices.

2.2. We consider HR-matrices $A_1, A_2, \ldots, A_s \in U(n)$ and put, for

$$x = (x_0, x_1, \ldots, x_s) \in \mathbf{R}^{s+1}$$

and $A_0 = E_n$ ($n \times n$ unit matrix)

$$f(x) = \sum_0^s x_j A_j .$$

For all x with $|x| = 1$, $f(x)$ is a unitary matrix: this is, as mentioned in the Introduction, precisely the meaning of the HR-matrix relations (1).

Let further $B_1, B_2, \ldots, B_t \in U(m)$ be HR-matrices, and for

$$y = (y_0, y_1, \ldots, y_t) \in \mathbf{R}^{t+1}, \quad B_0 = E_m,$$

$$g(y) = \sum_0^t y_k B_k ;$$

$g(y) \in U(m)$ for all y with $|y| = 1$. We define F by

$$F(x, y) = \begin{pmatrix} f(x) \otimes E_m & E_n \otimes g(y) \\ -E_n \otimes \overline{g(y)}^T & \overline{f(x)}^T \otimes E_m \end{pmatrix}.$$

One immediately checks that $F(x, y)\bar{F}^T(x, y) = (|x|^2 + |y|^2)E_{2nm}$. Thus $F(x, y) \in U(2nm)$ for all $(x, y) \in \mathbf{R}^{s+t+2}$ with $|x|^2 + |y|^2 = 1$. Since the coefficient matrix of x_0 is E_{2nm} the coefficient matrices of $x_1, \ldots, x_s, y_0, \ldots, y_t$ constitute a set of $s + t + 1$ HR-matrices $\in U(2nm)$. They are, explicitly,

$$(5) \quad \begin{pmatrix} A_j \otimes E_m & 0 \\ 0 & -A_j \otimes E_m \end{pmatrix}, \begin{pmatrix} 0 & E_{nm} \\ -E_{nm} & 0 \end{pmatrix}, \begin{pmatrix} 0 & E_n \otimes B_k \\ E_n \otimes B_k & 0 \end{pmatrix}$$

with $j = 1, \ldots, s$ and $k = 1, \ldots, t$. In other words, we have a product of ε-representations of G_s and G_t

$$D_s^U \times D_t^U \overset{\cup}{\to} D_{s+t+1}^U .$$

Since addition in D_s^U is by the direct sum of ε-representations this product is clearly distributive. Associativity (up to equivalence) is easily checked. We thus get a ring structure in $D_*^U = \bigoplus_{-1}^{\infty} D_s^U$; we have added the term $D_{-1}^U = \mathbf{Z}$ generated by the ring unit. The ring D_*^U is graded if the grading is by $s+1$ for D_s.

From the HR-matrices (5) of the product one notes that if one of the two factors is restricted from D_*^U so is the product; i.e., $h*D_*^U$ is a (graded) ideal in D_*^U, and we get a (graded) ring structure in $D_*^U/h*D_*^U = E_*^U$.

The same procedure yields, of course, a (graded) ring structure in $E_*^O = \bigoplus_{s=-1}^{\infty} E_s^O$, with grading $s+1$ for E_s^O. In 2.3 and 2.4 below these rings are described explicitly.

Remark 2.1. An easy computation shows that the rings E_*^U and E_*^O are anticommutative with respect to the grading, i.e., commutative except for the factor $(-1)^{(s+1)(t+1)}$. This will not really be used since the E_s^U and E_s^O are all 0, $\mathbf{Z}$ or $\mathbf{Z}/2$. We just note that in the case $\mathbf{Z}$, with generator ρ_s, $-\rho_s$ is given by the other equivalence class of irreducible ε-representations, see 2.1.

2.3. The ring E_*^U.

The generator ρ_s of E_s^U, given by an irreducible unitary ε-representation of G_s, has degree $2^{s/2}$ if s is even, $2^{(s-1)/2}$ if s is odd. The product $\rho_s\rho_t \in E_{s+t+1}^U$ has degree

$$
\begin{array}{lll}
2^{(s+t+2)/2} & \text{if} & s \text{ and } t \text{ are even}\,, \\
2^{(s+t+1)/2} & \text{if} & s \text{ is even, } t \text{ odd, or vice-versa}\,, \\
2^{(s+t)/2} & \text{if} & s \text{ and } t \text{ are odd}\,.
\end{array}
$$

Thus, unless both s and t are even, the product is irreducible, i.e., $\rho_s\rho_t = \pm\rho_{s+t+1}$. After choice of $\rho_1 \in E_1^U$ we can choose $\rho_3 = \rho_1^2$, $\rho_5 = \rho_1\rho_3 = \rho_3\rho_1 = \rho_1^3, \ldots$, and for all odd $s = 2r-1$, $\rho_s = \rho_1^r$; for even s, $E_s^U = 0$.

PROPOSITION 2.2. *The product with* $\rho_1 \in E_1^U$ *is an isomorphism* $E_s^U \cong E_{s+2}^U$ *for all* s. *For odd* $s = 2l-1$ *we choose*

$$\rho_{2l-1} = \rho_1^l\,,\ l = 1, 2, 3, \ldots\,.$$

THEOREM 2.3. *E_*^U is the polynomial ring* $\mathbf{Z}[\rho_1]$.

2.4. THE RING E_*^O.

We denote by σ_s the generator of E_s^O ($= 0$ if $s \equiv 2, 4, 5, 6$ modulo 8; determined up to sign if $s \equiv 3, 7$ modulo 8 where $E_s^O = \mathbf{Z}$).

The generator ρ_7 ($= \rho_1^4) \in E_7^U$ can be given by a real ε-representation of degree 8 which we can use as generator $\sigma_7 \in E_7^O$. The ring homomorphism $\Phi: E_*^O \to E_*^U$ induced by the embedding $O \to U$, $\Phi(\sigma_7) = \rho_7$, is thus an isomorphism $E_7^O \cong E_7^U$. In E_*^O the degree of $\sigma_7\sigma_s \in E_{s+8}^O$ is $16d_s^O = d_{s+8}^O$. Hence $\sigma_7\sigma_s$ is irreducible, i.e., $= \pm\, \sigma_{s+8}$ for all s. In particular we can choose $\sigma_{15} = \sigma_7^2, \sigma_{23} = \sigma_7^3, \dots, \sigma_{8r-1} = \sigma_7^r$.

PROPOSITION 2.4. *The isomorphism $E_s^O \cong E_{s+8}^O$ can be given by the product with $\sigma_7 \in E_7^O$.*

PROPOSITION 2.5. *$\sigma_7 \in E_7^O$ generates a subring of E_*^O which is the polynomial ring* $\mathbf{Z}[\sigma_7]$.

We further note that $\sigma_3 \in E_3^O$ is mapped by Φ to $2\rho_3 \in E_3^U$. From $\Phi(\sigma_3^2) = 4\rho_3^2 = 4\rho_7 = \Phi(4\sigma_7)$ we infer that $\sigma_3^2 = 4\sigma_7$. As for $\sigma_0 \in E_0^O$, it is of degree 1 and order 2, and $\sigma_0^2 \in E_1^O$ is of degree 2 and order 2, i.e., $\sigma_0^2 = \sigma_1$. Of course $\sigma_0^3 = 0$.

In summary:

THEOREM 2.6. *E_*^O is the commutative ring, graded by $s + 1$ for E_s^O, generated by $\sigma_0, \sigma_3, \sigma_7$ with the only relations $2\sigma_0 = 0$, $\sigma_0^3 = 0$, $\sigma_3^2 = 4\sigma_7$.*

3. THE HOMOTOPY GROUPS OF U AND O

3.1. We will deal explicitly with the unitary case. The orthogonal case can be treated in almost exactly the same way; any additional arguments will be mentioned wherever necessary.

In the Introduction 0.1 we associated with a set of s unitary $n \times n$ HR-matrices, i.e., with an ε-representation of G_s, a map $f: S^s \to U$ of the s-sphere $S^s \subset \mathbf{R}^{s+1}$ into the infinite unitary group U via $U(n)$. Since conjugation is homotopic to the identity, equivalent representations yield homotopic maps f (in the orthogonal case, we have to observe that conjugation can be made with a matrix from the identity component). The map $\phi: D_s^U \to \pi_s(U)$ thus obtained is a homomorphism; indeed, homotopy group addition of f and f' in $\pi_s(U(n))$ can be replaced by multiplication in