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LIE BRACKET AND CURVATURE

by Hans SAMELSON [1]

We consider two standard facts, which can be described briefly as (a) Lie bracket = infinitesimal commutator, and (b) Curvature = infinitesimal holonomy. The usual proofs of these facts use Taylor expansions in some form and run quite parallel to each other. It is our purpose to *deduce* (b) from (a); the point being that covariant differentiation, suitably interpreted, is a Lie bracket.

1. (a) Let M be a (smooth, C^∞) manifold (of dimension n), and consider two vectorfields X and Y on M (defined, say, as derivations of the **R**-algebra of smooth real-valued functions on M, or pointwise, i.e., as (smooth) sections of the tangent bundle TM of M, with X_p or $X(p)$ denoting the value at a point p of M). The Lie bracket $[XY]$ is then the operator $X \circ Y - Y \circ X$ on the algebra of functions, which happens to be a vector field again.

(b) Let E be a vector bundle over M (e.g., the tangent bundle), with projection $\pi: E \to M$, and let D be a connection on E (defined, say, as a function that assigns to each vectorfield X on M an operator D_X that sends any section s of E to another section $D_X s$, additive and satisfying (1) $D_{fX}s = f \cdot D_X s$ and (2) $D_X f \cdot s = f \cdot D_X s + Xf \cdot s$; alternatively, D assigns to each point of E a "horizontal" subspace h of the tangent space E_e to E at e, complementary to the tangent space to the fiber of π through e, with certain linearity conditions).

A standard simple calculation shows that for two vectorfields X and Y on M the operator $D_{[XY]} - D_X \circ D_Y + D_Y \circ D_X$ (which sends sections of E to sections of E) is in fact a tensor, a section of $\operatorname{Hom}(E, E)$, which at each point p of M defines a linear map of the fiber $\pi^{-1}(p) = E^p$ of E at p to itself. The tensor is denoted by R_{XY}, and called the curvature tensor of D; the map at p is denoted by $R_{XY}(p)$. The value $R_{XY}(p)$ depends only on the values X_p and Y_p of X and Y *at* P (and not on

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the values at other points) (and the curvature tensor has some additional properties which we don't need). For more detailed definitions one might consult [2].

2. Both Lie bracket $[XY]$ and curvature R_{XY} tensor are related to the flows $\exp(X, t)$ and $\exp(Y, t)$ of X and Y.

(a) For the bracket one constructs, for a given value of t, a map $\varphi(t): M \to M$ by, starting with any point p in M, following first the X-flow, then the Y-flow, then the $-X$-flow, finally the $-Y$-flow, each time from 0 to t; i.e., one applies the commutator

$$\exp(-Y, t) \circ \exp(-X, t) \circ \exp(Y, t) \circ \exp(X, t) .$$

(Thus one forms a "curved square", which however is usually not closed, i.e., one has $\varphi(t, p) \neq p$.) The fact "Lie bracket = infinitesimal commutator" mentioned in the Introduction is the following formula (including the existence of the limit on the right)

$$\text{(L)} \qquad [XY]_p = \lim_{t \to 0} \frac{1}{t^2} \big(\varphi(t, p) - p\big) .$$

Here the difference on the right is interpreted as taking place in $\mathbf{R}^n$, via any coordinate system at p. For a recent proof see [1].

(b) There is a similar development for the curvature. This time we take two vectorfields X, Y with $[XY] = 0$. It is standard fact that then the two flows described in (a) commute, and so $\varphi(t) = id$, and the "square" is now a closed curve, going from p back to p. Moving the points in the fiber E^p D-parallel around the square, one gets the holonomy transformation $H(t)$, which at each p gives a linear map $H(t, p)$ of the fiber E^p to itself. The fact "curvature = infinitesimal holonomy" mentioned in the Introduction is the following formula

$$\text{(H)} \qquad R_{XY}(p) = \lim_{t \to 0} \frac{1}{t^2} \big(H(t, p) - id\big) .$$

For a proof see again [1], e.g.

3. As noted above, the proofs for (L) and (H) are completely parallel. This situation, the same proof for two facts, has always seemed unsatisfactory to the writer. The purpose of this note is to derive (H) as an application of (L), by interpreting covariant derivative D as a Lie bracket (to be sure in E, not in M).

For each vectorfield X in M we define its horizontal lift X^h, a vector field on E, by defining the value $X^h(e)$ at any e in E to be the unique horizontal vector, in the horizontal space $h(e)$ at e, that projects to X_p under π (here $p=\pi(e)$). We note that the flow for X^h is D-parallel transport for E along X.

Similarly, for each section s of E we define its vertical extension s^v, a vector field in E, by assigning to any e in E the vertical vector $s(p)$ at e (one has to note that the fiber E^p, for $p = \pi(e)$, is a vector space and that therefore one has the standard identification of E^p with its own tangent space at any point). Thus s^v, restricted to a fiber, is a "constant" vectorfield in E^p, with value $s(p)$. Both X^h and s^v are π-projectable, in the sense of Chevalley, with X^h projecting to X and s^v projecting to 0.

4. The main observation now expresses covariant derivative as Lie bracket.

FACT. Let X be a vector field in M, and let s be a section of E. Then

$$(D_X s)^v = [X^h s^v] .$$

Interpreting the operation $[X^h -]$ as Lie derivative, i.e. as the infinitesimal action of the flow of X^h on tangent vectors to E, one sees easily that the right hand side is at any rate of the form s_1^v for some section s_1 of E: the flow for X^h, being D-parallel transport, sends a constant vector field in one fiber to constant fields in the transported fibers.

For the proof of the Fact: It is practically a tautology, if one interprets $D_X s$ as the (infinitesimal) deviation of s from being D-parallel along X. Or again: First suppose s is D-parallel along X. Then the flow for X^h maps s^v to itself, and as a result we have $[X^h s^v] = 0$, so the Fact checks in this case. Further, both sides of the equation in the Fact have the "derivation" property relative to functions f on M:

$$(D_X f \cdot s)^v = (f \cdot D_X s + Xt \cdot s)^v = f^v \cdot (D_X s)^v + (Xf)^v \cdot s^v$$

(where f^v means $f \circ \pi$, i.e. f pulled back to E), and

$$[X^h, f^v s^v] = f^v \cdot [X^h s^v] + X^h f^v \cdot s^v ;$$

clearly we have $(Xf)^v = X^h f^v$. Thus the Fact holds for fs, with s D-parallel along X. At any p in M with $X_p \neq 0$ there are sufficiently many such sections s_i to generate all sections as $\Sigma f_i s_i$. At zeros of X on the boundary of its zero-set the result follows by continuity; and at interior points it is trivial. (Incidentally, the right hand side is function-linear in X, since $s^v f^v = 0$ for any f; namely, f^v is constant on each fiber of E.)

5. Now to the proof of relation (H) in section 1, assuming (L).

Let X and Y be two vector fields in M with $[XY] = 0$; then the "square"-construction of section 2 (a) for X and Y on M has $\varphi(t) = id$ for all t. (Note that for a given p in M and vectors X_0, Y_0 at p we can arrange $X_p = X_0, Y_p = Y_0$.) To X and Y we form X^h and Y^h as in section 3. As already noted, the flows in E for X^h and for Y^h are D-parallel transport and map fibers of E linearly into fibers of E.

Thus the "square" construction of section 2 (a) for X^h and Y^h on E gives a map $H(t): E \to E$, which maps the fiber E^p at any p linearly to itself; this is the holonomy transformation. It follows that the right hand side in (L) (for X^h and Y^h on E) at each p gives a linear map, say $S_{XY}(p)$ of E^p to itself, which satisfies

$$[X^h Y^h](e) = S_{XY}(p)(e). \tag{S}$$

Here the right hand side has to be regarded as a tangent vector to E^p (and thus to E) at e (again using the usual identification for tangent spaces of vector spaces). (At this point the nature of the dependence of $S_{XY}(p)$ on X and Y is not clear).

To prove (H) we must show that $S_{XY}(p)$ equals $R_{XY}(p)$. The defining relation for R_{XY} is now $-D_X D_Y + D_Y D_X = R_{XY}$, because of $[XY] = 0$. Thus we must show

$$[X^h Y^h](s(p)) = ((-D_X D_Y + D_Y D_X)s)^v(s(p))$$

for any sections s of E and any p in M. (Recall that now we must regard $R_{XY}(p)(e)$ not as a point in E^p, but as a tangent vector to the vector space E^p at e.) By the Fact of section 4 the right hand side is

$$-[X^h[Y^h s^v]](s(p)) + [Y^h[X^h s^v]](s(p)).$$

By the Jacobi identity for vectorfields this equals

$$-[[X^h Y^h] s^v](s(p)).$$

We must show that this equals $[X^h Y^h](s(p))$.

Now the field $[X^h Y^h]$ is everywhere tangent to the fibers of E since it projects to $[XY] = 0$; also s^v is tangent to the fibers, by definition. Thus it is enough to evaluate everything on the individual fibers E^p.

And on each fiber E^p the field s^v is constant, and the field $[X^h Y^h]$ is linear (where a linear vectorfield P on a vector space V is defined by a linear map, also denoted by P, of V to itself, and assigns to a vector w the vector $P(w)$ *qua* tangent vector at w). It is elementary that for a linear vector field P, and a constant vectorfield Q with value w_0, on a vector space V the bracket $[PQ]$ is again constant, with value $-P(w_0)$. Thus the value of $[[X^h Yh] s^v]$ at any e in E^p is $-[X^h Y^h](s(p))$, and our result follows.

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