THE FIXED POINT SET OF A FINITE GROUP ACTION ON A HOMOLOGY FOUR SPHERE

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1. INTRODUCTION

It is a classical result of P. A. Smith that a finite p-group acting on a finite dimensional complex with the Z/p homology of a sphere S^k has fixed point set Z/p homologically equivalent to some S^k with $k < n$.

This theorem cannot, in general, be extended to groups of more general type, even if one assumes much more restrictive hypotheses such as a smooth action on a manifold homeomorphic or diffeomorphic to a sphere.

In particular, for every odd $n \geqslant 5$, it is possible to use Brieskorn varieties to produce finite group actions with fixed point set not a homology sphere. Even if the fixed point set is a sphere it can be embedded in a non standard way; for an elementary discussion of this phenomenon see [18] and [15]. For other "strange" actions of groups on higher dimensional spheres and disks the reader is referred to [16].

In low dimensions it is harder to construct such examples, and it may be conjectured that finite group actions on spheres are equivalent or somewhat "close" to linear ones.

On S^2 the situation is the best possible, indeed according to [6], [13] and [8], every finite group of homeomorphisms of S^2 is topologically conjugate to a linear action.

On S^3 it is necessary to assume local linearity, otherwise pathologies such as horned spheres may arise, see [2].

A deep and difficult theorem, conjectured by Smith and proved by combining results of Thurston, Meeks and Yau and Bass, states that every smooth cyclic action on S^3 is conjugate to the linear one; a detailed account can be found in [15].

The first example of a smooth cyclic action on S^4 with fixed point set a knotted S^2 is in [12], for more information see [9] and [17], these actions are obviously not linear.

The aim of this paper is to prove that any locally linear orientation preserving action of a finite group on an homology four sphere has fixed point set homeomorphic to a sphere. In particular there are no one fixed point actions. Besides, if the fixed point set is S^0, it is proved that the local representations are conjugate.[1]) For a large class of actions, the proof is an elementary application of Smith's theory, using the fact that in dimensions $\leqslant 2$ homology spheres are topological spheres. In one remaining case, an action of the icosahedral group, a slightly more complicated argument is needed. This type of argument cannot be extended to dimension 3, as the example in [11] proves.

The motivation for this work came from the paper of Peter Braam and Gordana Matic [3] on group actions and instantons spaces. They prove that a smooth orientation preserving action of a group on a homology sphere whose fundamental group has no nontrivial representations in $SU(2)$ admits an even number of isolated fixed points and that they come in pair such that the representations around them are conjugate. Also, Furuta proved that there are no actions with one fixed point.

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2. Statement of the result

In the following "R-homology S^n" will mean a compact topological manifold whose homology with coefficients in the ring R is the same as that of S^n. (Of course in dimensions 0, 1, 2 such a manifold is homeomorphic to a sphere.) To unify some notation, the empty set will be considered a sphere of dimension -1, all actions will be assumed effective.

THEOREM 2.1. *Let G be a finite group acting locally linearly and preserving the orientation on a Z-homology 4-sphere Σ. Then the fixed point set of G is homeomorphic to a sphere; in particular it never consists of one point.*

Local linearity is assumed to avoid pathologies, every smooth action is locally linear (see e.g. [5]).

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[1]) The author has been informed that this has been proved independently by S. Cappell.

Observation 2.2. There is a non-locally linear action of a finite group on a homology four sphere with exactly one fixed point.

Proof. Take the one fixed point action of A_5 on the Poincaré's sphere constructed in [11], remove the fixed point and multiply the remaining homology disk by the unit interval to obtain a four homology disk on which the product action has no fixed points. One can extend this action to the one point compactification to obtain a homology S^4 on which A_5 acts fixing only the point at infinity.

The main tool in the proof of Theorem 2.1. will be the classical result due to Smith (see [19]);

THEOREM 2.3. *Let Z/p, p a prime, act on a Z/p homology S^n, then the fixed point set is a Z/p homology S^k; if p is odd, $n - k$ is even.*

3. SOLVABLE GROUPS

In the four dimensional case it is easy to deduce from Theorem 2.3. the Corollary:

COROLLARY 3.1. *Let G be a solvable group acting locally linearly and orientation preserving on Σ, then the fixed point set is a sphere.*

Proof of the Corollary. Let $\{ I \} = H_0 \subset H_1 \subset H_2 \subset G$ be a composition series such that every H_{i+1} is normal in H_i and the quotients are cyclic of prime order p_i. By Smith theorem $X = \mathrm{Fix}(H_i)$ is a Z/p homology sphere, the action is not trivial so X cannot be the whole Σ; nor can it be 3-dimensional, for otherwise some element of H_1 would interchange the two components of $\Sigma - X$ and so reverse the orientation. Hence X has to be of dimension less than or equal to 2 and so a topological sphere.

For $i > 1$, $\mathrm{Fix}(H_{i-1})$ is invariant under H_i and the latter's action factorizes through H_i/H_{i-1}, so $\mathrm{Fix}(H_i) = \mathrm{Fix}\big(H_{i-1}/H_i \mid \mathrm{Fix}(H_{i-1})\big)$; applying repeatedly the argument above and using the fact that now all the spaces involved are spheres, the statement follows.

If $x_0 \in \Sigma^G$, the fixed set of G on Σ, the assumption of local linearity gives a representation $G \xrightarrow{\rho} SO(4)$, faithful since G acts effectively, this allows us to think of G as a finite subgroup of $SO(4)$ and to study it we look at the central extension:

3.2 $$0 \to C_2 \to SO(4) \to SO(3) \times SO(3) \to 0$$

where $\pi = (\pi_+, \pi_-)$ is given by the representation onto the self-dual and anti-self-dual forms in R^4, and C_2 is $\{\pm I\}$ the center of $SO(4)$.

Observe that $\pi^{-1}(\Delta)$, where Δ is the diagonal in $SO(3) \times SO(3)$, is the image of the "suspension" map from $O(3)$ into $SO(4)$:

$$M \to \begin{pmatrix} \det M & 0 \\ 0 & M \end{pmatrix}$$

We state now two elementary facts which will become useful in the following;

LEMMA 3.3. *If* $\alpha \in SO(4)$ *has at least one eigenvalue* $= 1$ *then its image* $\pi(\alpha) = (\alpha_+, \alpha_-)$ *in* $SO(3) \times SO(3)$ *is conjugate to an element of* Δ, *i.e.,* $v^{-1}\alpha_+ v = \alpha_-$ *for some* $v \in SO(3)$.

LEMMA 3.4. *The fixed space of an element of* $SO(4)$ *always has even dimension.*

Consider the diagram

3.5
$$\begin{array}{ccccccc} SO(4) & & \overset{\pi}{\to} & & SO(3) & \times & SO(3) \\ \cup & & & & \cup & & \cup \\ G \cdot C_2 = \tilde{G} & & \overset{\pi}{\to} & G_0 \subset & G_1 & \times & G_2 \\ j \cup & & & & & & \\ G & & & & & & \end{array}$$

where the G_is $(i=1, 2)$ are the images of the projections π_i of G_0 into the two $SO(3)$s; j is either the identity or the inclusion of a subgroup of index 2 in $\tilde{G} = \pi^{-1}(\pi(G))$ in the latter case $\pi \circ j$ appear as G_i. Luckily, finite subgroups of $SO(3)$ are well known (see e.g. [20]): they can be divided into four types:

i. cyclic groups C_n,

ii. dihedral groups D_{2m},

iii. the tetrahedral group,

iv. the octahedral group,

v. the icosahedral group.

All the first four types consist of solvable groups. It is easy to show that the class of solvable groups is closed under the operations of taking products, subgroups and central extensions, so G falls in the hypothesis of Corollary 3.1. in all cases, except the one in which at least one G_i is the icosahedral group. This is isomorphic to A_5, the alternating group on five letters and this identification will be fixed from now on.

4. Non solvable groups

We will prove Theorem 2.1 case by case. We start with the Lemma:

Lemma 4.1. *If G contains C_2, then* $\mathrm{Fix}(G)$ *is* S^0.

Proof. $\mathrm{Fix}(G) = \mathrm{Fix}\big(G/C_2 \mathrm{Fix}(C_2)\big)$. $\mathrm{Fix}(C_2)$ is a homology sphere by Smith's theorem and is zero dimensional since around the chosen fixed point the non trivial element of C_2 acts like the matrix $-I$, which has an isolated fixed point. The action of G/C_2 on S^0 has to be trivial since the fixed point set is required not to be empty.

By renumbering the factors and changing basis if necessary, we may assume G_2 equal to A_5, with $G_2 \xrightarrow{i} SO(3)$ the standard representation of A_5. Then G_0 is a subgroup of $G_1 \times A_5$ mapping onto both factors and to study it in more detail we look at the kernel of the second projection: $G_0 \xrightarrow{\pi_2} A_5$. This subgroup consists of elements of the form (k, I) with $k \in G_1$; we denote it by K_1.

For convenience we distinguish three cases:

Case 1. K_1 is a non-trivial subgroup of $SO(3)$, not isomorphic to A_5,

Case 2. K_1 is isomorphic to A_5,

Case 3. K_1 is trivial.

Proof in case 1. The surjection $G \to A_5$ has non trivial kernel $K = j^{-1}(\pi^{-1}(K_1)) \subset G$, this group is solvable since K_1 is, π is a central extension and j is an injection. By Corollary 3.1., $\mathrm{Fix}(K)$ is a sphere of dimension 2 and $\mathrm{Fix}(G)$ is the fixed point set of an A_5 acting on it, so it is easy to see that the only actions admitting some fixed points are the trivial ones.

Proof in case 2. Since A_5 is not properly contained in any finite subgroup of $SO(3)$, K_1 has to be equal to the whole G_1.

So $G_0 \subset A_5 \times A_5 \subset SO(3) \times SO(3)$ and contains $K_1 = A_5 \times \{I\}$, it follows that G_0 is the whole $A_5 \times A_5$. Observe that the two inclusions of A_5 in $SO(3)$ do not necessarily agree.

We claim that G in the diagram 3.5 must contain C_2, for if not $j \circ \pi$ would be an isomorphism $G \to A_5 \times A_5$ and its inverse would split the extension

$$0 \to C_2 \to \tilde{G} \to A_5 \to A_5 \to 0$$

This is not possible (see the appendix). Now apply Lemma 4.1. to end the proof.

Proof in case 3. If K_1 is trivial the projection $G_0 \xrightarrow{\pi_2} A_5$ is an isomorphism and the composition $\phi = \pi_1 \circ \pi_2^{-1} : A_5 \to G_1$ is a map onto, with graph G_0. The homomorphic images of A_5 are only the trivial group and A_5 itself, since A_5 is simple.

If $G_1 = \phi(A_5)$ is trivial, G_0 is equal to $\{I\} \times A_5$. As in case 2 the extension

$$0 \to C_2 \to G \to \{I\} \times A_5$$

is not split, so G contains C_2 and $\mathrm{Fix}(G) = S$ by 4.1. If $G_1 = \phi(A_5)$ is isomorphic to A_5, $G_0 \subset G_1 \times G_2$ is a copy of A_5 too, mapped into $SO(3) \times SO(3)$ according to $d(x) = (h(x); i(x))$, where $h(x)$ is some irreducible representation and $i(x)$ is the standard one specified before. The arguments in [22] can be used to prove that there are exactly two equivalence classes of representations of A_5 into $SO(3)$.

So there are two subcases:

a. h is $x \to u^{-1} i(x) u$, with $u \in SO(3)$,

b. h is conjugate to the composition $\bar{i}: A_5 \xrightarrow{\sigma} A_5 \xrightarrow{i} SO(3)$ and σ is conjugation by the cycle $(\bar{i}_2) S_5$ on A_5.

a. If the coordinate system around the fixed point chosen at the beginning is linearly changed according to some $\tilde{u} \in SO(4)$, the representation $\rho: G \to SO(4)$ becomes $\tilde{u}\rho(x)\tilde{u}^{-1}$.

If $\pi(\tilde{u}) = (u; 1)$; i is left unchanged and h is replaced by i. So G_0 is contained in the diagonal and $G \in \tilde{G} \in Im(O(3))$.

Recall that when G contains C_2, $\mathrm{Fix}(G) = S^0$ by Lemma 4.1.

LEMMA 4.2. *If* $G \neq C_2$, $\mathrm{Fix}(G) = S^1$.

Proof. G is isomorphic to A_5 and has to be contained in $Im(SO(3))$ so its representation has a one dimensional fixed space, which implies $\mathrm{Fix}(G)$ 1-dimensional at x_0. Now A_5 contains A_4 (named tetrahedral group when sitting in $SO(3)$), so $\mathrm{Fix}(A_5) \in \mathrm{Fix}(A_4)$, A_4 is solvable and hence

$\mathrm{Fix}(A_4)$ is a sphere. It cannot be S^2 since the representation of A_4 in $SO(3)$ is irreducible, so it is S^1. The only closed 1-dimensional submanifold of S^1 is S^1 itself, so $\mathrm{Fix}(G) = S^1$.

b. As in subcase a., a linear change in coordinates allows us to assume that h is actually $\tilde{i}$, and as before if $G_2 \in G$ the proposition is proved applying 4.1.

If it is not the case, let α correspond to the cycle $(12345) \in A_5$, β to (123) and γ to (345). We observe that β and γ generate A_5 and so:

1. $\mathrm{Fix}(A_5) = \mathrm{Fix}(\beta) \cap \mathrm{Fix}(\gamma)$,
2. $\mathrm{Fix}(A_5) \subset \mathrm{Fix}(\alpha)$.

We claim that $\mathrm{Fix}(\alpha)$ is S^0. According to Smith's theorem it is enough to prove that the representation of α around x_0 has an isolated fixed point, i.e. is the sum of two irreducible complex ones.

If not by Lemma 3.3 $(\bar{i}(\alpha); i(\alpha))$ would be conjugate in $SO(3) \times SO(3)$ to an element on the diagonal. From the explicit description of i *and* $\bar{i}$ (see the end of section 7.1 of [22]), it follows that they send all the five cycles to non conjugate elements in $SO(3)$, so this is impossible, and $\mathrm{Fix}(\alpha) = S^0$.

As for β and γ, their images under $(\bar{i}, i)$ are conjugate to elements on the diagonal, by 3.3 and 3.4 their fixed point sets have two-dimensional components, and so by Smith's theorem they are copies of S^2.

So $\mathrm{Fix}(G)$ is the intersection of a couple of S^2s and is contained in $\mathrm{Fix}(\alpha)$ which is S^0. If this set is empty or equal to S^0, the proposition follows. If it were a single point, it would be a transverse intersection, by local linearity, but it is not possible since a homology S^4 does not contain any two cycles with intersection number odd. This ends the proof.

5. Locally linear representation

Let's now consider the case of G acting on a homology S^4 with two fixed points, P_0 and P_1.

THEOREM 5.1. *The unoriented representations of* G *around* P_0 *and* P_1 *are linearly equivalent.* [1])

Proof. It will suffice to show that the characters associated to the representations around the P_is agree on every cyclic subgroup C_k of G.

[1]) See the note in the introduction.

Observe that by Lemma 3.4 and Smith's theorem the fixed point set of an element of G different from the identity is either S^0 or S^2.

Let g generate C_k, we distinguish three cases:

1. $\operatorname{Fix}(g^r) = \{P_1; P_2\}$ for every $r \equiv 0 (\operatorname{mod} k)$,
2. $\operatorname{Fix}(g) = S^2$,
3. $\operatorname{Fix}(g) = \{P_1; P_2\}$ but $\operatorname{Fix}(g^n) = S^2$ for some $g^n \neq id$.

Case 1. The hypothesis means that the action is semifree and the claim follows from the work of Atiyah and Bott, see [1] and [14].

Case 2. The action of C_k on the normal bundle of the fixed S^2 defines an element N of $K_{C_k}(S^2)$. Since C_k acts trivially on S^2 the two inclusions $P_i \to S^2$ are obviously C_k homotopic so that the diagram:

$$\begin{tikzcd}[column sep=small]
 & K_{C_k}(P_2) \arrow[dr] & \\
{[N] \in K_{C_k}(S^2)} \arrow[ur] \arrow[dr] & & R(C_k) \\
 & K_{C_k}(P_1) \arrow[ur] &
\end{tikzcd}$$

commutes. This means that the representation of C_k in the normal component to S^2 are conjugate, the tangential representations are of course both the identity, so the statement is proved.

Case 3. We can assume, by [8], that the action on $S^2 = \operatorname{Fix}(g^n)$ is linear. S^2 has zero intersection number in Σ so its normal bundle N can be identified to $S^2 \times R^2$, and we fix a trivialization. Denote a point of $S^2 - \{P_1; P_2\}$ by (x, t) with $x \in S^1$ and $t \in (0, 1)$. Let C_0 be the space $\{\phi : S^1 \to SO(2) \mid \deg \phi = 0\}$, it is an abelian group by pointwise multiplication and a C_k module with structure given by:

$$(h\phi)(x) = \phi(hx), h \in C_k \quad \text{and} \quad x \in S^1 \subset S^2$$

acted on by the obvious induced action.

By [5], chapter VI, prop. 11.1, the action is given by a θ_t such that

1. $\theta_t \in Z^1(C_k; C_0)$ and depends continuously on $t \in [0, 1]$.
2. $\theta_i(h)(x)$ is constant on $x \in S^1$ and equal to the representation of h at P_i for $i = 0; 1$.

A change in the trivialization adds to each θ_t a coboundary so there is a well defined continuous family $\theta_t : [0, 1] \to H^1(C_k; C_0)$.

A straightforward calculation shows that $H^1(C_k; C_0) = H^2(C_k; Z) = C_k$. Since θ_t is continuous it has to be constant, so $\theta_0 = \theta_1$ and by 2. the two normal representations are equal. In the topological case, by the results of Cappel and Shaneson topological equivalence of matrices in dimension 4 implies linear equivalence, so the statement of Theorem 5.1 makes sense also for a group of homeomorphism.

The proof given can be adapted to this more general case provided that the followings are true:

1. the topological Atiyah-Singer signature formula holds,
2. a locally flat S^2 in Σ has a normal bundle,
3. the argument in case 3 works with Homeo (S^1) instead of $SO(2)$.

Assertion 1 is proved, in the case of the semi-free action, in [21], page 188; assertion 2 follows from the work of Freedman, see [10]; assertion 3 is proved using the retraction Homeo (S^1) into $SO(2)$ given by the Poincaré number, see [7].

Appendix

LEMMA. *The extensions:*

$$\begin{array}{ccccccccc}
0 & \to & C_2 & \to & \tilde{A}_5 & \to & A_5 & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & C_2 & \to & A_5 \times A_5 & \to & A_5 \times A_5 & \to & 0 \\
 & & & & \downarrow & & \downarrow{\scriptstyle (h,h')} & & \\
0 & \to & C_2 & \to & SO(4) & \to & SO(3) \times SO(3) & \to & 0
\end{array}$$

are not split, h and h' can be any nontrivial representations of A_5 and f is either $(Id \times \{I\})$ or $(\{I\} \times Id)$.

Proof. Standard theory of group extensions and cohomology (see [4]) allows us to reduce to the:

PROPOSITION. *Any non trivial homomorphism $A_5 \xrightarrow{i} SO(3)$ induces an isomorphism $Z/2 = H^2(BSO(3); Z/2) \xrightarrow{i} H^2(BA_5; Z/2) = Z/2$.*

Proof of the Proposition. If the corresponding extension is split, then $Z/2 \times A_5 \subset S^3$, but $A_5 = 60$ so there exists a $Z/2 \subset A_5$ so $Z/2 \times Z/2$ would act freely on S^3, which cannot happen.

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