§3. Consequences of the Main Theorem

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At this point we simply count lattice points directly and construct the table below. The table shows $P(\sqrt{p}) > \frac{p-1}{4}$ for all primes $p < 100$ except $p = 5$. This completes the proof of Theorem 1.7 and hence the Main Theorem has been proved completely.

p	$P(\sqrt{p})$	Range of primes where $P(\sqrt{p}) > \frac{p-1}{4}$
59	27	$59 \leqslant p \leqslant 97$
31	15	$31 \leqslant p \leqslant 53$
19	9	$19 \leqslant p \leqslant 29$
11	5	$11 \leqslant p \leqslant 17$
7	3	$p = 7$
3	1	$p = 3$
5	1	$1 \not> \frac{5-1}{4}$

§ 3. Consequences of the Main Theorem

In this section we derive some consequences of the Main Theorem that have applications to the algebraic theory of quadratic forms. The results in this section are well known ([La], Chapter 6). The point is that we have new and more elementary proofs.

Let $\langle\langle a, b, c\rangle\rangle$ denote the 3-fold Pfister form

$$\langle 1, a\rangle \otimes \langle 1, b\rangle \otimes \langle 1, c\rangle = \langle 1, a, b, c, ab, ac, bc, abc\rangle .$$

3.1. Proposition. *Let $a, b, c \in \mathbf{Q}^\times$. Then $\langle\langle a, b, c\rangle\rangle$ is hyperbolic over $\mathbf{Q}$ if and only if a, b, c are not all positive.*

Proof. If $\langle\langle a, b, c\rangle\rangle$ is hyperbolic, then consideration of $\langle\langle a, b, c\rangle\rangle$ over the field of real numbers shows at least one of a, b, c is negative.

Now suppose $a < 0$. Then the Main Theorem implies $-c \in D_{\mathbf{Q}}(\langle\langle a, b\rangle\rangle)$ and $\langle\langle a, b\rangle\rangle \perp \langle c\rangle$ is isotropic over $\mathbf{Q}$ by Lemma 1.2 (b). A theorem of Pfister ([La], p. 279] implies $\langle\langle a, b, c\rangle\rangle$ is hyperbolic over $\mathbf{Q}$.

3.2. PROPOSITION. *Let* $a, b, c \in \mathbf{Q}^\times$, $a, b, c > 0$. *Then*

$$\langle\langle a, b, c\rangle\rangle \cong \langle\langle 1, 1, 1\rangle\rangle = 8\langle 1\rangle .$$

Proof. Calculating in the Witt ring WF we have

$$\begin{aligned}\langle\langle a, b, 1\rangle\rangle \perp (-1)\langle\langle a, b, c\rangle\rangle &= \langle\langle a, b\rangle\rangle (\langle 1, 1\rangle \perp (-1)\langle 1, c\rangle) \\ &= \langle\langle a, b\rangle\rangle \langle 1, -c\rangle = \langle\langle a, b, -c\rangle\rangle = 0 \text{ by Proposition 3.1.}\end{aligned}$$

Therefore $\langle\langle a, b, 1\rangle\rangle \cong \langle\langle a, b, c\rangle\rangle$. Repeating the same calculation with a, b in place of c yields the result.

3.3. COROLLARY. *Let* $a, b, c \in \mathbf{Q}^\times$ *and let* $\mathbf{H} = \langle 1, -1\rangle$. *Then*

$$\langle\langle a, b, c\rangle\rangle \cong \begin{cases} \langle\langle 1, 1, 1\rangle\rangle & \textit{if } a, b, c > 0 \\ 4\mathbf{H} & \textit{otherwise} . \end{cases}$$

3.4. THEOREM. $I^3\mathbf{Q}$ *is torsion-free.*

Proof. Corollary 3.3 shows that the only nonzero 3-fold Pfister form in $I^3\mathbf{Q}$ is $\langle\langle 1, 1, 1\rangle\rangle$. Therefore $I^3\mathbf{Q} \cong \mathbf{Z}$ and $I^3\mathbf{Q}$ is torsion-free.

REFERENCES

[An] ANDREWS, G. *Number Theory.* W.S. Saunders, Philadelphia, 1971.

[BS] BOREVICH, Z. I. and I. R. SHAFAREVICH. *Number Theory.* Academic Press, New York, 1966.

[Ca] CASSELS, J. W. S. *Rational Quadratic Forms.* Academic Press, New York, 1978.

[Ha] HASSE, H. *Vorlesungen Über Zahlentheorie.* Springer-Verlag, Berlin, 1964.

[HW] HARDY, G. H. and E. M. WRIGHT. *An Introduction to the Theory of Numbers.* Oxford University Press, 4th ed., 1960.

[La] LAM, T. Y. *The Algebraic Theory of Quadratic Forms.* Benjamin, 1973.

[Mo] MORDELL, L. J. *Diophantine Equations.* Academic Press, New York, 1969.

[Om] O'MEARA, O. T. *Introduction to Quadratic Forms.* Springer-Verlag, Berlin, 1963.

[Se] SERRE, J.-P. *A Course in Arithmetic.* Springer-Verlag, 1973.

[Sk] SKOLEM, Th. On the Diophantine Equation $ax^2 + by^2 + cz^2 + dv^2 = 0$. *Norske Videnskabers Selsk. Forh. 21* (1948), 76-79.

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