# 5. R-MATRICES AND INTERTWINING OPERATORS


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## 5. $R$-MATRICES AND INTERTWINING OPERATORS

In this section we shall prove that, after a trivial twisting, the intertwining operators between certain representations of Yangians provide rational solutions of the quantum Yang-Baxter equation. Recall that, if $V$ is any representation of $Y = Y(\mathfrak{sl}_2)$, then, for any $a \in \mathbf{C}$, we denote by $V(a)$ its pull-back by the automorphism $\tau_a$ of $Y$ defined in Proposition 2.5.

PROPOSITION 5.1. *Let* $V, W$ *be irreducible finite-dimensional representations of* $Y$ *with highest weight vectors* $\Omega_V, \Omega_W$ *and let* $a, b \in \mathbf{C}$. *Then:*

*(a) the tensor products* $V(a) \otimes W(b)$ *and* $W(b) \otimes V(a)$ *are irreducible and isomorphic except for a finite set of values* $S(V, W)$ *of* $a - b$;

*(b) the unique intertwining operator*

$$I(V, a; W, b): W(b) \otimes V(a) \to V(a) \otimes W(b)$$

*which maps* $\Omega_W \otimes \Omega_V$ *to* $\Omega_V \otimes \Omega_W$ *is a rational function of* $a - b$ *with values in* $\mathrm{Hom}(W \otimes V, V \otimes W)$.

*Proof.* Part (a) follows immediately from Proposition 4.2 and Corollary 4.7. For part (b), we need the following lemma.

LEMMA 5.2. *Let* $V, W$ *be representations of* $Y$ *and let* $a \in \mathbf{C}$.

*(a) If* $V$ *is irreducible, so is* $V(a)$.

*(b) If* $I: V \to W$ *is an isomorphism of representations of* $Y$, *so is* $I: V(a) \to W(a)$.

*Proof of lemma.* Part (a) follows from the definition of $V(a)$. For part (b), we must show that $I$ commutes with the action of $x$ and $J(x)$ on $V(a)$ and $W(a)$, for all $x \in \mathfrak{sl}_2$. But this is clear, since the action of $x$ is the same as that on $V$ and $W$, and that of $J(x)$ is the same as that of $J(x) + ax$ on $V$ and $W$.

Returning to the proof of Proposition 5.1, it follows from the lemma that $I(V, a; W, b)$ is a function of $a - b$, so it suffices to consider the case $b = 0$. For any $a \in \mathbf{C}$ which does not belong to the finite set $S(V, W)$, there is a unique isomorphism

$$I(V, a; W, 0) \equiv I(a): W \otimes V(a) \to V(a) \otimes W$$

of representations of $Y$ such that

$$I(a)\,(\Omega_W \otimes \Omega_V) = \Omega_V \otimes \Omega_W\,. \tag{5.3}$$

Choose bases of $V \otimes W$ and $W \otimes V$ and let $\{I_\lambda\}$ be a basis of $\mathfrak{sl}_2$; write $I(a)$ also for the matrix of $I(a)$ with respect to these bases. Let $A_\lambda, B_\lambda$ be the matrices of $I_\lambda$ and $J(I_\lambda)$ acting on $W \otimes V(a)$; and let $A'_\lambda$ and $B'_\lambda$ refer similarly to $V(a) \otimes W$. Then, $I(a)$ commutes with the action of $Y$ if and only if $I(a)$ satisfies the following system of homogeneous linear equations:

$$A_\lambda I(a) = I(a)A'_\lambda\,, \qquad B_\lambda I(a) = I(a)B'_\lambda\,, \qquad \text{for all} \quad \lambda\ .$$

We know that, if $a \notin S(V, W)$, these equations have a unique solution satisfying equation (5.3). By elementary linear algebra, the solution is a rational function of the entries of the matrices $A_\lambda, A'_\lambda, B_\lambda, B'_\lambda$. Since $A_\lambda, A'_\lambda$ are independent of $a$ and $B_\lambda, B'_\lambda$ are linear in $a$, the result follows.

*Definition 5.4.* Let $V$ be a finite-dimensional irreducible representation of $Y$. Then, the *R-matrix associated to* $V$ is the function $R(a-b)$ with values in $\operatorname{End}(V \otimes V)$ given by

$$R(a-b) = I(V, a; V, b)\sigma\ ,$$

where $\sigma \in \operatorname{End}(V \otimes V)$ is the switch of the two factors.

THEOREM 5.5. *Let $V$ be a finite-dimensional irreducible representation of $Y$. Then the R-matrix associated to $V$ is a rational solution of the quantum Yang-Baxter equation:*

$$R^{12}(a-b)R^{13}(a-c)R^{23}(b-c) = R^{23}(b-c)R^{13}(a-c)R^{12}(a-b)\ . \tag{5.6}$$

*Proof.* We note first some simple commutation relations between the intertwining operator $I(a-b) \equiv I(V, a; V, b)$ and the switch map $\sigma$. For example, we have

$$\sigma^{12}I^{13}(a-c)\sigma^{12} = I^{23}(a-c)\ .$$

by an easy computation. Similarly,

$$\sigma^{12}\sigma^{13}I^{23}(b-c)\sigma^{13}\sigma^{12} = I^{12}(b-c)\ .$$

Hence,

$$\begin{aligned} R^{12}(a-b)R^{13}(a-c)R^{23}(b-c) &= I^{12}(a-b)\sigma^{12}I^{13}(a-c)\sigma^{13}I^{23}(b-c)\sigma^{23} \\ &= I^{12}(a-b)I^{23}(a-c)\sigma^{12}\sigma^{13}I^{23}(b-c)\sigma^{23} \\ &= I^{12}(a-b)I^{23}(a-c)I^{12}(b-c)\sigma^{12}\sigma^{13}\sigma^{23}\ . \end{aligned}$$

Similarly,

$$R^{23}(b-c)R^{13}(a-c)R^{12}(a-b) = I^{23}(b-c)I^{12}(a-c)I^{23}(a-b)\sigma^{23}\sigma^{13}\sigma^{12}\ .$$

Hence, in view of the relation

$$\sigma^{12}\sigma^{13}\sigma^{23} = \sigma^{23}\sigma^{13}\sigma^{12}$$

in the symmetric group on three letters, the equation to be proved is

$$I^{12}(a-b)I^{23}(a-c)I^{12}(b-c) = I^{23}(b-c)I^{12}(a-c)I^{23}(a-b) \ . \tag{5.7}$$

Note that both sides of equation (5.7) define intertwining operators

$$V(c) \otimes V(b) \otimes V(a) \to V(a) \otimes V(b) \otimes V(c)$$

which fix the tensor product of the highest weight vectors in $V$. Hence, regarded as functions on $\mathbf{C}^3$ with values in $\operatorname{End}(V \otimes V \otimes V)$, they agree on the complement of the set $S$ of $(a, b, c) \in \mathbf{C}^3$ where $V(c) \otimes V(b) \otimes V(a)$ or $V(a) \otimes V(b) \otimes V(c)$ is reducible. It follows from part (a) of Proposition 5.1 that $S$ intersects each complex line parallel to one of the axes in $\mathbf{C}^3$ in at most finitely many points. It is easy to see that the complement of such a set is Zariski dense in $\mathbf{C}^3$. Since the two sides of equation (5.7) are rational functions which agree on a Zariski dense set, they are equal.

*Remark.* We have used the following simple fact about intertwining operators. Let $U$, $V$ and $W$ be representations of a Yangian $Y(\mathfrak{sl}_2)$ and let $I: U \otimes V \to V \otimes U$ be an intertwining operator. Then

$$I^{12}: U \otimes V \otimes W \to V \otimes U \otimes W$$

and

$$I^{23}: W \otimes U \otimes V \to W \otimes V \otimes U$$

are intertwining operators. While this is obvious enough, it should be noted that

$$I^{13}: U \otimes W \otimes V \to V \otimes W \otimes U$$

is *not* an intertwining operator in general.

We conclude this general discussion by showing that, up to a sign change in the parameter, the $R$-matrix $R(u)$ we have associated to a representation of $Y$ is the same as that constructed using the "universal $R$-matrix" (see Theorem 3 of [4]). Set

$$\tilde{R}(u) = R(-u) \ .$$

Then, by Theorem 4 of [4], it suffices to prove that

$$P_\lambda^+(a, b)\tilde{R}(b-a) = \tilde{R}(b-a)P_\lambda^-(a, b) \tag{5.8}$$

where

$$P_\lambda^\pm(a, b) = (\rho \otimes \rho)\left((J(I_\lambda) + aI_\lambda) \otimes 1 + 1 \otimes (J(I_\lambda) + bI_\lambda) + \frac{1}{2}[I_\lambda \otimes 1, \Omega]\right),$$

$\rho: Y \to \mathrm{End}(V)$ is the action of $Y$ on $V$ and $\{I_\lambda\}$ is an orthonormal basis of $\mathfrak{sl}_2$. In terms of intertwining operators, equation (5.8) asserts that

$$P_\lambda^+(a, b)I(a-b) = I(a-b)\sigma P_\lambda^-(a, b)\sigma .$$

But it is easy to see that

$$\sigma P_\lambda^-(a, b)\sigma = P_\lambda^+(b, a) .$$

Hence, we must prove that

$$P_\lambda^+(a, b)I(a-b) = I(a-b)P_\lambda^+(b, a) .$$

But this is simply the statement that

$$I(a-b): V(b) \otimes V(a) \to V(a) \otimes V(b)$$

commutes with the action of $J(I_\lambda)$.

We shall now apply these results to compute the $R$-matrices associated to every finite-dimensional irreducible representation of $Y$. By Theorem 4.11, every such representation is of the form

$$V = V_{m_1}(a_1) \otimes \cdots \otimes V_{m_k}(a_k).$$

The intertwining operator

$$I(a-b): V(b) \otimes V(a) \to V(a) \otimes V(b)$$

can be computed as the product of $k^2$ intertwining operators of the form $I(V_m, a; V_n, b)$, each of which effects an interchange of nearest neighbours. Since such an operator commutes, in particular, with the action of $\mathfrak{sl}_2$, it can be written in the form

$$I(V_m, a; V_n, b) = \sum_{j=0}^{\min\{m,n\}} c_j P_{m+n-2j} , \tag{5.9}$$

where

$$P_{m+n-2j}: V_n \otimes V_m \to V_m \otimes V_n$$

is the projection onto the irreducible component of

$$V_m \otimes V_n \cong \otimes_{j=0}^{\min\{m,n\}} V_{m+n-2j}$$

of type $V_{m+n-2j}$. We have $c_0 = 1$ since $I(V_m, a; V_n, b)$ preserves the tensor products of the highest weight vectors.

To compute $I(V_m, a; V_n, b)$, let $\Omega_j, j = 0, 1, \ldots, \min\{m, n\}$, be a highest weight vector in $V_n \otimes V_m$ of weight $m+n-2j$; then, the vector $\Omega_j'$ obtained by switching the order of the factors in $\Omega_j$ is a highest weight vector in $V_m \otimes V_n$ of the same weight, and we have

$$I(V_m, a; V_n, b)\,(\Omega_j) = \Omega_j' .$$

Further, it is easy to see that, for $j > 0, (x^+ \otimes 1).\Omega_j$ is an $\mathfrak{sl}_2$-highest weight vector of weight $m+n-2j+2$; it is non-zero, since otherwise $\Omega_j$ would be annihilated by $x^+ \otimes 1$ and by $1 \otimes x^+$, contracting the assumption $j > 0$. Hence, we may assume that

$$(x^+ \otimes 1).\Omega_j = \Omega_{j-1}$$

for $j > 0$. Switching the order of the factors, we have

$$(x^+ \otimes 1).\Omega_j' = -\Omega_{j-1}' .$$

By Proposition 4.2 (and its proof), $\Omega_j$ is a $Y$-highest weight vector in $V_n(b) \otimes V_m(a)$ if

$$b - a = \frac{1}{2}(m+n) - j + 1 .$$

It follows from the formula for the co-multiplication in Definition 1.1 that, in the representation $V_n(b) \otimes V_m(a)$,

$$J(x^+).\Omega_j = \left(b - a - \frac{1}{2}(m+n) + j - 1\right)(x^+ \otimes 1).\Omega_j ,$$

and that in the representation $V_m(a) \otimes V_n(b)$,

$$J(x^+).\Omega_j' = \left(a - b - \frac{1}{2}(m+n) + j - 1\right)(x^+ \otimes 1).\Omega_j' .$$

The equation

$$I(V_m, a; V_n, b)\,(J(x^+).\Omega_j) = J(x^+).(I(V_m, a; V_n, b)\Omega_j)$$

now gives

$$\frac{c_j}{c_{j-1}} = \frac{a - b + \frac{1}{2}(m+n) - j + 1}{a - b - \frac{1}{2}(m+n) - j + 1} .$$

It follows that

$$(5.10) \qquad I(V_m, a; V_n, b) = \sum_{j-0}^{\min\{m,n\}} \prod_{i=0}^{j=1} \frac{a-b+\frac{1}{2}(m+n)-i}{a-b-\frac{1}{2}(m+n)+i} P_j .$$

We summarize our results in the following theorem.

THEOREM 5.11. *The R-matrix associated to the representation*

$$V = V_{m_1}(a_1) \otimes \cdots \otimes V_{m_k}(a_k)$$

*of* $Y$ *is given by*

$$R(a-b) = \Big(\prod_{i,j=1}^{k} I(V_{m_i}, a + a_i; V_{m_j}, b + a_j)\Big)\sigma ,$$

*where the intertwining operators are given by equation (5.10) and* $\sigma$ *is the switch map. The order of the factors in the product is such that the* $(i, j)$*-term appears to the left of the* $(i', j')$*-term iff*

$$i > i' \quad \textit{or} \quad i = i' \quad \textit{and} \quad j < j' .$$

## 6. CONCLUDING REMARKS

Since we have discussed only the Yangian associated to $\mathfrak{sl}_2$ in this paper, it may be worth-while to indicate the extent to which the results above can be generalized to the Yangian $Y(\mathfrak{a})$ associated to an arbitrary finite-dimensional complex simple Lie algebra $\mathfrak{a}$.

The definition of $Y(\mathfrak{a})$ is precisely as in (1.1), except of course that $\{I_\lambda\}$ should be an orthonormal basis of $\mathfrak{a}$ with respect to some invariant inner product. The formulae

$$\tau_a(x) = x , \qquad \tau_a(J(x)) = J(x) + ax ,$$

for $x \in \mathfrak{a}$, again define a one-parameter group of Hopf algebra automorphisms of $Y(\mathfrak{a})$, and the relation, discussed in section 5, between solutions of the quantum Yang-Baxter equation and intertwining operators between tensor products of representations of $Y(\mathfrak{a})$, which follows from the existence of the $\tau_a$, is also valid in the general case.