

### **3. The homomorphism**

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **36 (1990)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

#### **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

#### **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

by Proposition 3, we have  $J = \rho I$ , where  $\psi = \frac{p\phi + q}{r\phi + s}$ ,  $\rho = \frac{A}{a} \frac{1}{r\phi + s} = \varepsilon(r\bar{\phi} + s)$  and  $\varepsilon = ps - qr = \pm 1$ . Clearly we have

$$\begin{aligned} A &= \varepsilon a(r\phi + s)(r\bar{\phi} + s) = \varepsilon(as^2 + bsr - cr^2), \\ B &= A(\psi + \bar{\psi}) = \varepsilon a(\psi + \bar{\psi})(r\phi + s)(r\bar{\phi} + s) \\ &= \varepsilon a((p\phi + q)(r\bar{\phi} + s) + (p\bar{\phi} + q)(r\phi + s)) \\ &= \varepsilon(2asq + b(sp + rq) - 2cpr), \\ -C &= A\psi\bar{\psi} = \varepsilon a\psi\bar{\psi}(r\phi + s)(r\bar{\phi} + s) = \varepsilon a(p\phi + q)(p\bar{\phi} + q) \\ &= \varepsilon(aq^2 + bqp - cp^2). \end{aligned}$$

Thus  $A, B, C$  are integral linear combinations of  $a, b, c$ . Similarly,  $a, b, c$  are integral linear combinations of  $A, B, C$ . Hence  $\text{GCD}(A, B, C) = \text{GCD}(a, b, c) = 1$  so that  $J$  is primitive.

### 3. THE HOMOMORPHISM $\theta$

Let  $O_D$  and  $O_{D'}$  be two orders of  $O_{D_0}$  with  $O_{D'} \subset O_D$ . Then we have  $D' = Df^2$  for some positive integer  $f$ . This notation will be used throughout the rest of the paper. Our aim is to define a surjective homomorphism  $\theta$  from the ideal class group  $C_{D'}$  onto the ideal class group  $C_D$ . After proving three lemmas, we will prove the following theorem.

**THEOREM 1.** (i) Every class  $C$  of  $C_{D'}$  contains a primitive ideal  $I$  of the form  $I = \left[ a, \frac{fb + \sqrt{D'}}{2} \right]$ , where  $\text{GCD}(a, f) = 1$ , such that the ideal  $J = \left[ a, \frac{b + \sqrt{D}}{2} \right]$  is a primitive ideal of  $O_D$ .

(ii) If  $I = \left[ a, \frac{fb + \sqrt{D'}}{2} \right]$  ( $\text{GCD}(a, f) = 1$ ) and  $I' = \left[ a', \frac{fb' + \sqrt{D'}}{2} \right]$  ( $\text{GCD}(a', f) = 1$ ) are two primitive ideals in the same class  $C$  of  $C_{D'}$  with  $I' = \rho I$  ( $\rho \in K^*$ ), then the ideals

$$J = \left[ a, \frac{b + \sqrt{D}}{2} \right] \quad \text{and} \quad J' = \left[ a', \frac{b' + \sqrt{D}}{2} \right]$$

of  $O_D$  satisfy  $J' = \rho J$  and are in the same class  $\theta(C)$  of  $C_D$ .

(iii) The mapping  $C \rightarrow \theta(C)$  is a homomorphism of  $C_{D'}$  to on  $C_D$ .

Part (ii) of Theorem 1 will be the main tool in relating distances between ideals of different orders of the same real quadratic field.

LEMMA 1. A primitive ideal  $I = \left[ a, \frac{b + \sqrt{D}}{2} \right]$  contains a number  $\alpha = xa + y \left( \frac{b + \sqrt{D}}{2} \right)$ , where  $x$  and  $y$  are coprime integers, such that the integer  $N(\alpha)/a$  is prime to a given nonzero integer  $m$ .

*Proof.* We begin by noting that  $\frac{1}{a} N\left(xa + y \left( \frac{b + \sqrt{D}}{2} \right)\right) = ax^2 + bxy - cy^2$  in view of (2.5). If  $|m| = 1$  we take  $x = 1$ ,  $y = 0$ ,  $\alpha = xa + y \left( \frac{b + \sqrt{D}}{2} \right) = a$ , so that  $GCD(N(\alpha)/a, m) = GCD(a, 1) = 1$ , as required.

Hence we may suppose that  $|m| > 1$ . Let  $p_i (i = 1, 2, \dots, n)$  be the distinct prime factors of  $m$ . For  $i = 1, 2, \dots, n$  we set

$$(x_i, y_i) = \begin{cases} (1, 0), & \text{if } p_i \nmid a, \\ (0, 1), & \text{if } p_i \mid a, \quad p_i \nmid c, \\ (1, 1), & \text{if } p_i \mid a, \quad p_i \mid c, \end{cases}$$

so that  $p_i \nmid ax_i^2 + bx_iy_i - cy_i^2$ . Let  $x'$  and  $y'$  be integers such that  $x' \equiv x_i \pmod{p_i}$  and  $y' \equiv y_i \pmod{p_i}$  for  $i = 1, 2, \dots, n$ , so that  $GCD(ax'^2 + bx'y' - cy'^2, m) = 1$ . The required number  $\alpha$  is given by  $\alpha = xa + y \left( \frac{b + \sqrt{D}}{2} \right)$ , where  $x = \frac{x'}{GCD(x', y')}$ ,  $y = \frac{y'}{GCD(x', y')}$ .

LEMMA 2. Let  $m$  be a given nonzero integer. Every class  $C$  of  $C_D$  contains a primitive ideal  $\left[ a, \frac{b + \sqrt{D}}{2} \right]$  with  $GCD(a, m) = 1$ .

*Proof.* Let  $\left[ a', \frac{b' + \sqrt{D}}{2} \right]$  be a primitive ideal of the class  $C$ . By Lemma 1 there exist coprime integers  $x$  and  $y$  such that

$$(3.1) \quad GCD(a'x^2 + b'xy - c'y^2, m) = 1.$$

Set  $\alpha = a'x^2 + b'xy - c'y^2$  and let  $r$  and  $s$  be integers such that  $xs - yr = 1$ . Next set

$$(3.2) \quad \rho = x + \left( \frac{b' - \sqrt{D}}{2a'} \right) y, \quad b = 2a'xr + b'(xs + yr) - 2c'ys,$$

so that

$$a = \rho \left( xa' + y \left( \frac{b' + \sqrt{D}}{2} \right) \right)$$

and

$$\frac{b + \sqrt{D}}{2} = \rho \left( ra' + s \left( \frac{b' + \sqrt{D}}{2} \right) \right).$$

Then we have

$$\begin{aligned} \left[ a, \frac{b + \sqrt{D}}{2} \right] &= \rho \left[ xa' + y \left( \frac{b' + \sqrt{D}}{2} \right), ra' + s \left( \frac{b' + \sqrt{D}}{2} \right) \right] \\ &= \rho \left[ a', \frac{b' + \sqrt{D}}{2} \right] \end{aligned}$$

so that  $\left[ a, \frac{b + \sqrt{D}}{2} \right]$  is an ideal equivalent to the primitive ideal  $\left[ a', \frac{b' + \sqrt{D}}{2} \right]$ . Hence, by Corollary 3,  $\left[ a, \frac{b + \sqrt{D}}{2} \right]$  is primitive.

**LEMMA 3.** *Let  $C$  and  $C'$  be two classes of  $C_D$ . Then there exist primitive ideals  $I = \left[ a, \frac{B + \sqrt{D}}{2} \right] \in C$  and  $I' = \left[ a', \frac{B + \sqrt{D}}{2} \right] \in C'$  with  $GCD(a, a') = 1$ . Moreover the ideal  $II'$  is primitive and  $II' = \left[ aa', \frac{B + \sqrt{D}}{2} \right]$ .*

*Proof.* By Lemma 2 there exist primitive ideals  $I = \left[ a, \frac{b + \sqrt{D}}{2} \right] \in C$  and  $I' = \left[ a', \frac{b' + \sqrt{D}}{2} \right] \in C'$  with  $GCD(a, a') = 1$ . As  $b \equiv D \equiv b' \pmod{2}$  and  $GCD(a, a') = 1$  there are integers  $k$  and  $k'$  such that  $k'a' - ka = \frac{b - b'}{2}$ .

Set  $B = b + 2ka = b' + 2k'a'$  so that

$$I = \left[ a, \frac{B + \sqrt{D}}{2} \right] \quad \text{and} \quad I' = \left[ a', \frac{B + \sqrt{D}}{2} \right].$$

Now  $D - B^2$  is divisible by both  $4a$  and  $4a'$ , and so, as  $\text{GCD}(a, a') = 1$ ,  $D - B^2$  is a multiple of  $4aa'$ , so that  $c'' = \frac{D - B^2}{4aa'} \in \mathbb{Z}$ . Hence  $\left[ aa', \frac{B + \sqrt{D}}{2} \right]$

is an ideal of  $O_D$  and we have

$$\begin{aligned} II' &= \left( aa', a \left( \frac{B + \sqrt{D}}{2} \right), a' \left( \frac{B + \sqrt{D}}{2} \right), \left( \frac{B + \sqrt{D}}{2} \right)^2 \right) \\ &= \left( aa', \frac{B + \sqrt{D}}{2} \right) \\ &= \left[ aa', \frac{B + \sqrt{D}}{2} \right]. \end{aligned}$$

Finally, any prime divisor of  $aa', B, c''$  must divide  $\text{GCD}(a, B, a'c'') = 1$  or  $\text{GCD}(a', B, ac'') = 1$ , as  $\text{GCD}(a, a') = 1$ , which is impossible. Hence the ideal  $II'$  is primitive.

We are now ready to prove Theorem 1.

*Proof of Theorem 1.* (i) By Lemma 2 the class  $C$  contains a primitive ideal  $I = \left[ a, \frac{b' + \sqrt{D'}}{2} \right]$  with  $\text{GCD}(a, f) = 1$ . Let  $k$  be an integer such that

$$\begin{cases} 2ak \equiv -b' \pmod{f}, & \text{if } f \equiv 1 \pmod{2}, \\ ak \equiv -\frac{b'}{2} + D \frac{f}{2} \pmod{f}, & \text{if } f \equiv 0 \pmod{2}, \end{cases}$$

and set  $b = (b' + 2ak)/f$ , so that  $I = \left[ a, \frac{fb + \sqrt{D'}}{2} \right]$ . As  $I$  is an ideal of  $O_{D'}$ ,  $(D' - f^2b^2)/4a$  is an integer, and so, as  $\text{GCD}(a, f) = 1$ ,  $c = (D - b^2)/4a$  is also an integer, showing that  $J = \left[ a, \frac{b + \sqrt{D}}{2} \right]$  is an ideal of  $O_D$ . Further, as  $I$  is primitive, we have  $\text{GCD}(a, bf, cf^2) = 1$ , and so  $\text{GCD}(a, b, c) = 1$ , showing that  $J$  is primitive.

(ii) If  $I' = \rho I$ , by Proposition 3, there exist integers  $p, q, r, s$  with  $ps - qr = \pm 1$  such that

$$(3.3) \quad \frac{fb' + \sqrt{D'}}{2a'} = \frac{p \left( \frac{fb + \sqrt{D'}}{2a} \right) + q}{r \left( \frac{fb + \sqrt{D'}}{2a} \right) + s}, \quad \rho = \pm \left( r \left( \frac{fb - \sqrt{D'}}{2a} \right) + s \right).$$

Rearranging the first equation in (3.3), we obtain the following equality among elements of  $O_D$

$$f\left(\frac{b' + \sqrt{D}}{2}\right) \left(rf\left(\frac{b + \sqrt{D}}{2}\right) + sa\right) = a' \left(pf\left(\frac{b + \sqrt{D}}{2}\right) + qa\right),$$

from which we deduce that  $f \mid qaa'$ . As  $\text{GCD}(aa', f) = 1$  there exists an integer  $q'$  such that  $q = q'f$ , so (3.3) can be rewritten as

$$\frac{b' + \sqrt{D}}{2a'} = \frac{p}{rf} \frac{\left(\frac{b + \sqrt{D}}{2a}\right) + q'}{\left(\frac{b + \sqrt{D}}{2a}\right) + s}, \quad p = \pm \left( rf \left(\frac{b - \sqrt{D}}{2a}\right) + s \right).$$

which, by Proposition 3, shows that  $J' = \rho J$ .

(iii) Let  $C \in C_{D'}$  and  $C' \in C_{D'}$ . By Lemma 2 and (i), we can choose an ideal  $I = \left[a, f\left(\frac{b + \sqrt{D}}{2}\right)\right]$  in  $C$  with  $\text{GCD}(a, f) = 1$  and then an ideal  $I' = \left[a', f\left(\frac{b' + \sqrt{D}}{2}\right)\right]$  in  $C'$  with  $\text{GCD}(a', af) = 1$ . By (i)  $\left[a, \frac{b + \sqrt{D}}{2}\right]$  and  $\left[a', \frac{b' + \sqrt{D}}{2}\right]$  are ideals of  $O_D$  and so we have  $b \equiv b' \pmod{2}$ . We choose integers  $K'$  and  $K$  such that  $K'a' - Ka = \frac{b - b'}{2}$ , and set  $B = b + 2Ka = b' + 2K'a'$ , so that  $I = \left[a, f\left(\frac{B + \sqrt{D}}{2}\right)\right]$  and  $I' = \left[a', f\left(\frac{B + \sqrt{D}}{2}\right)\right]$ . By Lemma 3 we see that  $II' = \left[aa', f\left(\frac{B + \sqrt{D}}{2}\right)\right]$  is a primitive ideal of the class  $CC'$ . But the primitive ideals  $J = \left[a, \frac{B + \sqrt{D}}{2}\right]$ ,  $J' = \left[a', \frac{B + \sqrt{D}}{2}\right]$ ,  $J'' = \left[aa', \frac{B + \sqrt{D}}{2}\right]$  belong respectively to the classes  $\theta(C)$ ,  $\theta(C')$ ,  $\theta(CC')$ , and, as  $JJ' = J''$  by Lemma 3, we have  $\theta(C)\theta(C') = \theta(CC')$ , showing that  $\theta$  is a homomorphism:  $C_{D'} \rightarrow C_D$ .

Finally we show that  $\theta$  is surjective. Let  $C$  be a class of  $C_D$  and let  $J = \left[a, \frac{b + \sqrt{D}}{2}\right]$  be a primitive ideal of  $C$  with  $\text{GCD}(a, f) = 1$  (Lemma 2).

Then we have  $GCD(a, b, c) = 1$ , where  $\frac{D - b^2}{4a} = c$ , and so  $GCD(a, bf, cf^2) = 1$ ,

showing that  $I = \left[ a, f \left( \frac{b + \sqrt{D}}{2} \right) \right]$  is a primitive ideal of  $O_{D'}$ . Hence  $C$  is the image of the class of  $I$  under  $\theta$ .

**COROLLARY 4.** *If the class  $C$  of  $O_{D'}$  contains the primitive ideal  $I = \left[ a, \frac{b + \sqrt{D'}}{2} \right]$ , where  $f^2 | a$ , then  $f | b$  and the class  $\theta(C)$  contains the primitive ideal  $J = \left[ \frac{a}{f^2}, \frac{\frac{b}{f} + \sqrt{D}}{2} \right]$  of  $O_D$ .*

*Proof.* As  $D' = Df^2 = b^2 + 4ac$ , and  $f^2 | a$ , we see that  $f | b$ , and so  $GCD(f, c) = 1$ . By Corollary 2 we have  $I = \left( \frac{\sqrt{D'} - b}{2a} \right) \left[ c, \frac{-b + \sqrt{D'}}{2} \right]$  and so, by Theorem 1, we see that  $\left[ c, \frac{-\frac{b}{f} + \sqrt{D}}{2} \right] \in \theta(C)$ . Finally, by Corollary 2,  $J = \left[ \frac{a}{f^2}, \frac{b/f + \sqrt{D}}{2} \right] = \frac{\left( \sqrt{D} + \frac{b}{f} \right)}{2c} \left[ c, \frac{-\frac{b}{f} + \sqrt{D}}{2} \right]$ , showing that  $J \in \theta(C)$ .

#### 4. REDUCED IDEALS

From now on in this paper we suppose that  $D_0 > 0$  so that we are only considering ideals in orders of a real quadratic field. An ideal  $I$  of  $O_D$  can be written in the form  $I = ad[1, \phi]$ , where  $\phi = \frac{b + \sqrt{D}}{2a}$ . By Proposition 1 (ii), if  $I = a'd'[1, \phi']$  is another representation of  $I$ , then  $a' = \pm a$  and  $\phi' \equiv \frac{a}{a'} \phi \pmod{1}$ . A real number of the form  $\frac{b + \sqrt{D}}{2a}$ , where  $c = \frac{D - b^2}{4a}$  is an integer and  $GCD(a, b, c) = 1$  is called a quadratic irrationality of discriminant  $D$ .