1. Extensions of connections

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where the sum runs over all intersections U of $i+1$ distinct elements of $\mathscr{U}$. Let $\check{\delta}: C^i(\mathscr{U}, \mathscr{S}^j) \to C^{i+1}(\mathscr{U}, \mathscr{S}^j)$ be the Čech co-boundary. We also have boundaries $d: C^i(\mathscr{U}, \mathscr{S}^j) \to C^i(\mathscr{U}, \mathscr{S}^{j+1})$.

Now let

$$C^n(\mathscr{U}, \mathscr{S}^{\cdot}) = \oplus\, C^p(\mathscr{U}, \mathscr{S}^q)$$

where the sum runs over $p+q=n$. For $c \in C^n(\mathscr{U}, \mathscr{S}^{\cdot})$, we let $c^{p,q}$ denote its p, q-th component. The hyper-coboundary

$$\partial: C^n(\mathscr{U}, \mathscr{S}^{\cdot}) \to C^{n+1}(\mathscr{U}, \mathscr{S}^{\cdot})$$

is defined as follows: For $c \in C^n(\mathscr{U}, \mathscr{S}^{\cdot})$, we set

$$(\partial c)^{p,q} = dc^{p-1,q} + (-1)^{p-1}\check{\delta} c^{p,q-1}\ .$$

Then the hypercohomology of $\mathscr{S}$ with respect to $\mathscr{L}$, $\mathbf{H}^{\cdot}(S, \mathscr{S}^{\cdot}, \mathscr{L})$, is defined to be $\operatorname{Ker}(\partial)/\operatorname{Image}(\partial)$ and $\mathbf{H}^{\cdot}(S, \mathscr{S}^{\cdot})$ is defined to be an appropriate limit of these groups over all ordered covers. In particular, if S is a scheme, $\mathscr{S}^{\cdot}$ is a complex of coherent sheaves and $\mathscr{L}$ is an affine open cover, then $\mathbf{H}^{\cdot}(S, \mathscr{S}^{\cdot})$ is naturally isomorphic to $\mathbf{H}^{\cdot}(S, \mathscr{S}^{\cdot}, \mathscr{L})$. If in addition S is affine $\mathbf{H}^{\cdot}(S, \mathscr{S}^{\cdot}) \cong H^{\cdot}(\Gamma(\mathscr{S}^{\cdot}))$.

1. Extensions of connections

Let S be smooth connected scheme over a field K of characteristic zero. Suppose (H, ∇_H) and (G, ∇_G) are integrable connections on S. The set of isomorphism classes of integrable extensions of (H, ∇_H) by (G, ∇_G) forms a group under Baer sum which we will call $\operatorname{Ext}(H, G)$.

Proposition 1.1.1. $\operatorname{Ext}(H, G) \cong H^1(G \otimes \check{H}, \nabla_G \otimes \check{\nabla}_H)$.

Proof. Since ∇_H is integrable, H is locally free. Let $\mathscr{L}$ be an ordered affine open cover of S such that $H(U)$ is a free $\mathscr{O}_S(U)$-module for each $U \in \mathscr{L}$. Suppose we have an extension

$$0 \to (G, \nabla_G) \to (E, \nabla) \to (H, \nabla_H) \to 0$$

of connections. Let $U \in \mathscr{L}$. Since $H(U)$ is free, there exists an $\mathscr{O}_S(U)$-module section $s_U: H(U) \to E(U)$. Now let $h_U = \nabla \circ s_U - s_U \circ \nabla_H$. We claim that h_U is an $\mathscr{O}_S(U)$-module homomorphism from $H(U)$ into $\Omega^1_S \otimes G(U)$, i.e. an element of $\operatorname{Hom}_{\mathscr{O}_S}(H, \Omega^1_S \otimes G)(U)$. Indeed, for $f \in \mathscr{O}_S(U)$ and $v \in H(U)$,

$$h_U(fv) = \nabla(s_U(fv)) - s_U(\nabla_H(fv)) = \nabla(fs_U(v)) - s_U(df \otimes v + f\nabla_H v)$$
$$= df \otimes s_U(v) - f\nabla(s_U(v)) - (df \otimes s_U(v) + fs_U(\nabla_H v)) = fh_U(v) .$$

Let $s_{U,V} = s_U - s_V \in \mathrm{Hom}_{\mathscr{O}_S}(H, G)(U \cap V)$. We claim that $(\{h_U\}, \{s_{U,V}\})$ is a hyper one-cocycle for the complex $(\Omega^{\bullet}_S \otimes \mathrm{Hom}_{\mathscr{O}_S}(H, G), \nabla_{H,G})$. First it is clear that $\{s_{U,V}\}$ is a one-cocycle for the sheaf $\mathrm{Hom}_{\mathscr{O}_S}(H, G)$. Second

$$\nabla_G \circ s_{U,V} - s_{U,V} \circ \nabla_H = \nabla \circ (s_U - s_V) - (s_U - s_V) \circ \nabla_H = h_U - h_V .$$

Finally, since

$$\nabla \circ \nabla \circ s_U = \nabla \circ s_U \circ \nabla_H + \nabla \circ h_U = h_U \circ \nabla_H + \nabla_G \circ h_U = \nabla_{H,G}(h_U) ,$$

(using Lemma 1.0.1) ∇ is integrable iff $\nabla_{H,G}(h) = 0$.

Moreover, suppose $\{s'_U\}$ is another collection of sections

$$s'_U : H(U) \to E(U),\ h'_U = \nabla' \circ s'_U - s'_U \circ \nabla$$

and $s'_{U,V} = s'_U - s'_V$. Then $r_U = s'_U - s_U \in \mathrm{Hom}_{\mathscr{O}_S}(H, G)$ and

$$h'_U = h + \nabla \circ r_U - r_U \circ \nabla_H = h + \nabla_G \circ r_U - r_U \circ \nabla_H = h + \nabla_{H,G}(r_U) .$$

And so $(\{h_U\}, \{s_{U,V}\}) - (\{h'_U\}, \{s'_{U,V}\})$ is the hyper-boundary of $\{r_U\}$. Thus we get a natural map from

$$\mathrm{Ext}(H, \mathscr{O}_X) \quad \text{into} \quad H^1(\mathrm{Hom}_{\mathscr{O}_S}(H, G), \nabla_{H,G}) \cong H^1(G \otimes \check{H}, \nabla_G \otimes \check{\nabla}_H) .$$

It is easy to see that this map is a homomorphism.

We can make a map back as follows. Given a hyper-cocycle $(\{h_U\}, \{s_{U,V}\})$ for the complex $(\Omega^{\bullet}_S \otimes \mathrm{Hom}_{\mathscr{O}_S}(H, G), \nabla_{H,G})$, let E be the sheaf determined by the condition that $E(U) = G(U) \oplus H(U)$ with gluing data

$$(w, v) \to (w + s_{U,V}, v)$$

on $U \cap V$. We then put a connection ∇ on E by setting

$$\nabla(w, v) = (\nabla_G w + h_U(v), \nabla_H v)$$

for local sections w and v of G and H on U. One can check easily that E is an extension of H by G and that this construction gives the inverse to the map above. $\square$

COROLLARY 1.1.2. *$\mathrm{Ext}(H, \mathscr{O}_S)$ is a K vector space and hence is uniquely divisible.*

COROLLARY 1.1.3. *Suppose S is affine and S' is a non-empty affine open of S. Then $\mathrm{Ext}(H, \mathscr{O}_S)$ injects into $\mathrm{Ext}(H \otimes \mathscr{O}_{S'}, \mathscr{O}_{S'})$.*

We note that taking duals yields an isomorphism between $\mathrm{Ext}(G, H)$ and $\mathrm{Ext}(\check{H}, \check{G})$. Also, upon identifying $(\check{G})^{\vee}$ with G, $\check{\nabla}_{\check{G}} = \nabla_G$.

LEMMA 1.1.4. *The diagram*

$$\begin{array}{ccc} \mathrm{Ext}(H, G) & \to & H^1(G \otimes \check{H}, \nabla_G \otimes \check{\nabla}_H) \\ \downarrow & & \downarrow \\ \mathrm{Ext}(\check{G}, \check{H}) & \to & H^1(\check{H} \otimes G, \check{\nabla}_H \otimes \nabla_G) \end{array}$$

anti-commutes, where the horizontal arrows are the isomorphisms given by the proposition and the right vertical arrow is the evident one.

Proof. Since the assertion is local, we may suppose H and G are free. Suppose (E, ∇) is an extension of H by G and $s: H \to E$ is a section. Then $h = \nabla \circ s - s \circ \nabla_H$ is an element of $\mathrm{Hom}_{\mathcal{O}_S}(H, \Omega_S^1 \otimes G)$ which represents the image of the isomorphism class of E in

$$H^1(\mathrm{Hom}(H, G), \nabla_{H,G}) \cong H^1(G \otimes \check{H}, \nabla_G \otimes \check{\nabla}_H)\ .$$

The image k of h in $\mathrm{Hom}_{\mathcal{O}_S}(\check{G}, \Omega_S^1 \otimes \check{H})$ is determined by

$$k(w)(v) = w(h(v)) = w((\nabla \circ s - s \circ \nabla_H)(v))$$

where v is a section of H and w is a section of $\check{G}$.

Now $(\check{E}, \check{\nabla})$ is an extension of $\check{G}$ by $\check{H}$ and the homomorphism t determined by

$$t(w)(e) = w(e - s \circ \pi(e))$$

is a section, where $\pi: E \to H$ is the projection, e is a section of E and w is a section of $\check{G}$. Hence, $g = \check{\nabla} \circ t - t \circ \nabla_{\check{G}}$ is an element of $\mathrm{Hom}_{\mathcal{O}_S}(\check{G}, \Omega_S^1 \otimes \check{H})$ which represents the image of the isomorphism class of $\check{E}$ in

$$H^1(\mathrm{Hom}(\check{G}, \check{H}), \nabla_{\check{G},\check{H}})\ .$$

Now

$$g(w)(v) = (\check{\nabla} \circ t - t \circ \nabla_{\check{G}})(w)(e)$$

where $e = s(v)$ and

$$\begin{aligned} \check{\nabla} \circ t(w)(e) &= d(w(e - s(\pi(e)) - w(\nabla(e) - s(\pi(\nabla(e)))) \\ &= -\, w(\nabla \circ s(v) - s \circ \nabla_H(v)) = -\, k(w)(v) \end{aligned}$$

since $\pi(s(v)) = v$ and $\pi\nabla(e) = \nabla_H(\pi(e))$. The lemma now follows from

$$(t \circ \nabla_G^\vee)(w)(e) = \nabla_G^\vee(w)(e - s(\pi(e)) = 0 . \quad \square$$

Suppose W is an $\mathcal{O}_S$ submodule of H. We let [W] denote the smallest subconnection of H containing W.

2. The Gauss-Manin connection

Here we will recall the definition and some basic properties of the Gauss-Manin connection which we will need in this paper. For more details see [K-O]. If $\mathcal{S}^\cdot$ is a complex, $\mathcal{S}^\cdot(k)$ will denote the complex obtained from $\mathcal{S}^\cdot$ by setting $\mathcal{S}^i(k) = \mathcal{S}^{i+k}$. For any scheme Y over K will let $K[Y]$ denote $\Gamma(\mathcal{O}_Y)$.

Suppose S is a smooth connected affine scheme over K. Suppose $f: X \to S$ is a smooth morphism, Z is a closed subscheme of X, smooth over S. Suppose T is either $\mathrm{Spec}(K)$ or S. Then we define the subcomplex $\Omega^\cdot_{X/T,Z}$ of $\Omega^\cdot_{X/T}$ by the exactness of the sequence.

$$0 \to \Omega^\cdot_{X/T,Z} \to \Omega^\cdot_{X/T} \to \Omega^\cdot_{Z/T} \to 0 .$$

When $T = \mathrm{Spec}(K)$ we drop it from the notation. It follows that $\Omega^i_{X/S,Z} = \Omega^i_{X/S}$ for $i > \dim_S Z$. Note that $\Omega^0_{X,Z} = \Omega^0_{X/S,Z}$ is the sheaf of ideals of Z on X. We define $H^i_{DR}(X/S, Z)$ to be the i-th hypercohomology group of the complex $\Omega^\cdot_{X/S,Z}$. We set $H^i_{DR}(X/S) = H^i_{DR}(X/S, \emptyset)$. If X is affine, then $H^i_{DR}(X/S, Z)$ is the i-th cohomology group of the complex of $K[S]$ modules $\Gamma(\Omega^\cdot_{X/S,Z})$. If X is affine, K has characteristic zero and U is a dense open subscheme of X then the natural map from $H^i_{DR}(X/S, Z)$ to $H^i_{DR}(U/S, U \cap Z)$ is an injection.

From the last short exact sequence with $T = S$, we obtain a long exact sequence

(2.1) $$\ldots \to H^{i-1}_{DR}(Z/S) \to H^i_{DR}(X/S, Z) \to H^i_{DR}(X/S) \to \ldots$$

The Gauss-Manin connection $\nabla: H^i_{DR}(X/S, Z) \to \Omega^1_S \otimes H^i_{DR}(X/S, Z)$ is the boundary map in the long exact sequence obtained by taking hypercohomology of the short exact sequence of complexes:

(2.2) $$0 \to f^*\Omega^1_S \otimes \Omega^\cdot_{X/S,Z}(-1) \to \Omega^\cdot_{X/S,Z}/f^*\Omega^2_S \otimes \Omega^\cdot_X(-2) \to \Omega^\cdot_{X/S,Z} \to 0$$

(which is exact because X and Z are smooth over S). It is an integrable connection. If K has characteristic zero and f is surjective and has geometrically connected fibers, then $H^0_{DR}(X/S) = K[S]$ and the Gauss-Manin