

# 1. Extensions of connections

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **36 (1990)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

## Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

## Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*

ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, [www.library.ethz.ch](http://www.library.ethz.ch)

<http://www.e-periodica.ch>

where the sum runs over all intersections  $U$  of  $i + 1$  distinct elements of  $\mathcal{U}$ . Let  $\check{\delta}: C^i(\mathcal{U}, \mathcal{S}^j) \rightarrow C^{i+1}(\mathcal{U}, \mathcal{S}^j)$  be the Čech co-boundary. We also have boundaries  $d: C^i(\mathcal{U}, \mathcal{S}^j) \rightarrow C^i(\mathcal{U}, \mathcal{S}^{j+1})$ .

Now let

$$C^n(\mathcal{U}, \mathcal{S}^\bullet) = \bigoplus C^p(\mathcal{U}, \mathcal{S}^q)$$

where the sum runs over  $p + q = n$ . For  $c \in C^n(\mathcal{U}, \mathcal{S}^\bullet)$ , we let  $c^{p,q}$  denote its  $p, q$ -th component. The hyper-coboundary

$$\partial: C^n(\mathcal{U}, \mathcal{S}^\bullet) \rightarrow C^{n+1}(\mathcal{U}, \mathcal{S}^\bullet)$$

is defined as follows: For  $c \in C^n(\mathcal{U}, \mathcal{S}^\bullet)$ , we set

$$(\partial c)^{p,q} = dc^{p-1,q} + (-1)^{p-1} \check{\delta} c^{p,q-1}.$$

Then the hypercohomology of  $\mathcal{S}$  with respect to  $\mathcal{L}, \mathbf{H}^\bullet(S, \mathcal{S}^\bullet, \mathcal{L})$ , is defined to be  $\text{Ker}(\partial)/\text{Image}(\partial)$  and  $\mathbf{H}^\bullet(S, \mathcal{S}^\bullet)$  is defined to be an appropriate limit of these groups over all ordered covers. In particular, if  $S$  is a scheme,  $\mathcal{S}^\bullet$  is a complex of coherent sheaves and  $\mathcal{L}$  is an affine open cover, then  $\mathbf{H}^\bullet(S, \mathcal{S}^\bullet)$  is naturally isomorphic to  $\mathbf{H}^\bullet(S, \mathcal{S}^\bullet, \mathcal{L})$ . If in addition  $S$  is affine  $\mathbf{H}^\bullet(S, \mathcal{S}^\bullet) \cong H^\bullet(\Gamma(\mathcal{S}^\bullet))$ .

## 1. EXTENSIONS OF CONNECTIONS

Let  $S$  be smooth connected scheme over a field  $K$  of characteristic zero. Suppose  $(H, \nabla_H)$  and  $(G, \nabla_G)$  are integrable connections on  $S$ . The set of isomorphism classes of integrable extensions of  $(H, \nabla_H)$  by  $(G, \nabla_G)$  forms a group under Baer sum which we will call  $\text{Ext}(H, G)$ .

**PROPOSITION 1.1.1.**  $\text{Ext}(H, G) \cong H^1(G \otimes \check{H}, \nabla_G \otimes \check{\nabla}_H)$ .

*Proof.* Since  $\nabla_H$  is integrable,  $H$  is locally free. Let  $\mathcal{L}$  be an ordered affine open cover of  $S$  such that  $H(U)$  is a free  $\mathcal{O}_S(U)$ -module for each  $U \in \mathcal{L}$ . Suppose we have an extension

$$0 \rightarrow (G, \nabla_G) \rightarrow (E, \nabla) \rightarrow (H, \nabla_H) \rightarrow 0$$

of connections. Let  $U \in \mathcal{L}$ . Since  $H(U)$  is free, there exists an  $\mathcal{O}_S(U)$ -module section  $s_U: H(U) \rightarrow E(U)$ . Now let  $h_U = \nabla \circ s_U - s_U \circ \nabla_H$ . We claim that  $h_U$  is an  $\mathcal{O}_S(U)$ -module homomorphism from  $H(U)$  into  $\Omega_S^1 \otimes G(U)$ , i.e. an element of  $\text{Hom}_{\mathcal{O}_S}(H, \Omega_S^1 \otimes G)(U)$ . Indeed, for  $f \in \mathcal{O}_S(U)$  and  $v \in H(U)$ ,

$$\begin{aligned} h_U(fv) &= \nabla(s_U(fv)) - s_U(\nabla_H(fv)) = \nabla(fs_U(v)) - s_U(df \otimes v + f\nabla_H v) \\ &= df \otimes s_U(v) - f\nabla(s_U(v)) - (df \otimes s_U(v) + fs_U(\nabla_H v)) = fh_U(v). \end{aligned}$$

Let  $s_{U,V} = s_U - s_V \in \text{Hom}_{\mathcal{O}_S}(H, G)(U \cap V)$ . We claim that  $(\{h_U\}, \{s_{U,V}\})$  is a hyper one-cocycle for the complex  $(\Omega_S^\cdot \otimes \text{Hom}_{\mathcal{O}_S}(H, G), \nabla_{H,G})$ . First it is clear that  $\{s_{U,V}\}$  is a one-cocycle for the sheaf  $\text{Hom}_{\mathcal{O}_S}(H, G)$ . Second

$$\nabla_G \circ s_{U,V} - s_{U,V} \circ \nabla_H = \nabla \circ (s_U - s_V) - (s_U - s_V) \circ \nabla_H = h_U - h_V.$$

Finally, since

$$\nabla \circ \nabla \circ s_U = \nabla \circ s_U \circ \nabla_H + \nabla \circ h_U = h_U \circ \nabla_H + \nabla_G \circ h_U = \nabla_{H,G}(h_U),$$

(using Lemma 1.0.1)  $\nabla$  is integrable iff  $\nabla_{H,G}(h) = 0$ .

Moreover, suppose  $\{s'_U\}$  is another collection of sections

$$s'_U : H(U) \rightarrow E(U), \quad h'_U = \nabla' \circ s'_U - s'_U \circ \nabla$$

and  $s'_{U,V} = s'_U - s'_V$ . Then  $r_U = s'_U - s_U \in \text{Hom}_{\mathcal{O}_S}(H, G)$  and

$$h'_U = h + \nabla \circ r_U - r_U \circ \nabla_H = h + \nabla_G \circ r_U - r_U \circ \nabla_H = h + \nabla_{H,G}(r_U).$$

And so  $(\{h_U\}, \{s_{U,V}\}) - (\{h'_U\}, \{s'_{U,V}\})$  is the hyper-boundary of  $\{r_U\}$ . Thus we get a natural map from

$$\text{Ext}(H, \mathcal{O}_X) \text{ into } H^1(\text{Hom}_{\mathcal{O}_S}(H, G), \nabla_{H,G}) \cong H^1(G \otimes \check{H}, \nabla_G \otimes \check{\nabla}_H).$$

It is easy to see that this map is a homomorphism.

We can make a map back as follows. Given a hyper-cocycle  $(\{h_U\}, \{s_{U,V}\})$  for the complex  $(\Omega_S^\cdot \otimes \text{Hom}_{\mathcal{O}_S}(H, G), \nabla_{H,G})$ , let  $E$  be the sheaf determined by the condition that  $E(U) = G(U) \oplus H(U)$  with gluing data

$$(w, v) \mapsto (w + s_{U,V}, v)$$

on  $U \cap V$ . We then put a connection  $\nabla$  on  $E$  by setting

$$\nabla(w, v) = (\nabla_G w + h_U(v), \nabla_H v)$$

for local sections  $w$  and  $v$  of  $G$  and  $H$  on  $U$ . One can check easily that  $E$  is an extension of  $H$  by  $G$  and that this construction gives the inverse to the map above.  $\square$

**COROLLARY 1.1.2.**  *$\text{Ext}(H, \mathcal{O}_S)$  is a  $K$  vector space and hence is uniquely divisible.*

**COROLLARY 1.1.3.** *Suppose  $S$  is affine and  $S'$  is a non-empty affine open of  $S$ . Then  $\text{Ext}(H, \mathcal{O}_S)$  injects into  $\text{Ext}(H \otimes \mathcal{O}_{S'}, \mathcal{O}_{S'})$ .*

We note that taking duals yields an isomorphism between  $\text{Ext}(G, H)$  and  $\text{Ext}(\check{H}, \check{G})$ . Also, upon identifying  $(\check{G})^\vee$  with  $G$ ,  $\check{\nabla}_G^\vee = \nabla_G$ .

LEMMA 1.1.4. *The diagram*

$$\begin{array}{ccc} \text{Ext}(H, G) & \rightarrow & H^1(G \otimes \check{H}, \nabla_G \otimes \check{\nabla}_H) \\ \downarrow & & \downarrow \\ \text{Ext}(\check{G}, \check{H}) & \rightarrow & H^1(\check{H} \otimes G, \check{\nabla}_H \otimes \nabla_G) \end{array}$$

*anti-commutes, where the horizontal arrows are the isomorphisms given by the proposition and the right vertical arrow is the evident one.*

*Proof.* Since the assertion is local, we may suppose  $H$  and  $G$  are free. Suppose  $(E, \nabla)$  is an extension of  $H$  by  $G$  and  $s: H \rightarrow E$  is a section. Then  $h = \nabla \circ s - s \circ \nabla_H$  is an element of  $\text{Hom}_{\mathcal{P}_S}(H, \Omega_S^1 \otimes G)$  which represents the image of the isomorphism class of  $E$  in

$$H^1(\text{Hom}(H, G), \nabla_{H, G}) \cong H^1(G \otimes \check{H}, \nabla_G \otimes \check{\nabla}_H).$$

The image  $k$  of  $h$  in  $\text{Hom}_{\mathcal{P}_S}(\check{G}, \Omega_S^1 \otimes \check{H})$  is determined by

$$k(w)(v) = w(h(v)) = w((\nabla \circ s - s \circ \nabla_H)(v))$$

where  $v$  is a section of  $H$  and  $w$  is a section of  $\check{G}$ .

Now  $(\check{E}, \check{\nabla})$  is an extension of  $\check{G}$  by  $\check{H}$  and the homomorphism  $t$  determined by

$$t(w)(e) = w(e - s \circ \pi(e))$$

is a section, where  $\pi: E \rightarrow H$  is the projection,  $e$  is a section of  $E$  and  $w$  is a section of  $\check{G}$ . Hence,  $g = \check{\nabla} \circ t - t \circ \nabla_G^\vee$  is an element of  $\text{Hom}_{\mathcal{P}_S}(\check{G}, \Omega_S^1 \otimes \check{H})$  which represents the image of the isomorphism class of  $\check{E}$  in

$$H^1(\text{Hom}(\check{G}, \check{H}), \nabla_{\check{G}, \check{H}}^\vee).$$

Now

$$g(w)(v) = (\check{\nabla} \circ t - t \circ \nabla_G^\vee)(w)(e)$$

where  $e = s(v)$  and

$$\begin{aligned} \check{\nabla} \circ t(w)(e) &= d(w(e - s(\pi(e))) - w(\nabla(e) - s(\pi(\nabla(e))))) \\ &= -w(\nabla \circ s(v) - s \circ \nabla_H(v)) = -k(w)(v) \end{aligned}$$

since  $\pi(s(v)) = v$  and  $\pi\nabla(e) = \nabla_H(\pi(e))$ . The lemma now follows from

$$(t \circ \nabla_G^\vee)(w)(e) = \nabla_G^\vee(w)(e - s(\pi(e))) = 0. \quad \square$$

Suppose  $W$  is an  $\mathcal{O}_S$  submodule of  $H$ . We let  $[W]$  denote the smallest subconnection of  $H$  containing  $W$ .

## 2. THE GAUSS-MANIN CONNECTION

Here we will recall the definition and some basic properties of the Gauss-Manin connection which we will need in this paper. For more details see [K-O]. If  $\mathcal{S}^\cdot$  is a complex,  $\mathcal{S}^\cdot(k)$  will denote the complex obtained from  $\mathcal{S}^\cdot$  by setting  $\mathcal{S}^i(k) = \mathcal{S}^{i+k}$ . For any scheme  $Y$  over  $K$  will let  $K[Y]$  denote  $\Gamma(\mathcal{O}_Y)$ .

Suppose  $S$  is a smooth connected affine scheme over  $K$ . Suppose  $f: X \rightarrow S$  is a smooth morphism,  $Z$  is a closed subscheme of  $X$ , smooth over  $S$ . Suppose  $T$  is either  $\text{Spec}(K)$  or  $S$ . Then we define the subcomplex  $\Omega_{X/T, Z}^\cdot$  of  $\Omega_{X/T}^\cdot$  by the exactness of the sequence.

$$0 \rightarrow \Omega_{X/T, Z}^\cdot \rightarrow \Omega_{X/T}^\cdot \rightarrow \Omega_{Z/T}^\cdot \rightarrow 0.$$

When  $T = \text{Spec}(K)$  we drop it from the notation. It follows that  $\Omega_{X/S, Z}^i = \Omega_{X/S}^i$  for  $i > \dim_S Z$ . Note that  $\Omega_{X, Z}^0 = \Omega_{X/S, Z}^0$  is the sheaf of ideals of  $Z$  on  $X$ . We define  $H_{DR}^i(X/S, Z)$  to be the  $i$ -th hypercohomology group of the complex  $\Omega_{X/S, Z}^\cdot$ . We set  $H_{DR}^i(X/S) = H_{DR}^i(X/S, \emptyset)$ . If  $X$  is affine, then  $H_{DR}^i(X/S, Z)$  is the  $i$ -th cohomology group of the complex of  $K[S]$  modules  $\Gamma(\Omega_{X/S, Z}^\cdot)$ . If  $X$  is affine,  $K$  has characteristic zero and  $U$  is a dense open subscheme of  $X$  then the natural map from  $H_{DR}^i(X/S, Z)$  to  $H_{DR}^i(U/S, U \cap Z)$  is an injection.

From the last short exact sequence with  $T = S$ , we obtain a long exact sequence

$$(2.1) \quad \dots \rightarrow H_{DR}^{i-1}(Z/S) \rightarrow H_{DR}^i(X/S, Z) \rightarrow H_{DR}^i(X/S) \rightarrow \dots$$

The Gauss-Manin connection  $\nabla: H_{DR}^i(X/S, Z) \rightarrow \Omega_S^1 \otimes H_{DR}^i(X/S, Z)$  is the boundary map in the long exact sequence obtained by taking hypercohomology of the short exact sequence of complexes:

$$(2.2) \quad 0 \rightarrow f^*\Omega_S^1 \otimes \Omega_{X/S, Z}^\cdot(-1) \rightarrow \Omega_{X/S, Z}^\cdot / f^*\Omega_S^2 \otimes \Omega_X^\cdot(-2) \rightarrow \Omega_{X/S, Z}^\cdot \rightarrow 0$$

(which is exact because  $X$  and  $Z$  are smooth over  $S$ ). It is an integrable connection. If  $K$  has characteristic zero and  $f$  is surjective and has geometrically connected fibers, then  $H_{DR}^0(X/S) = K[S]$  and the Gauss-Manin