4. Abelian schemes

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **36 (1990)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

$$(\{\bar{\omega}_U - d_{X/S}g_U\}, \{f_{U,V} - (g_U - g_V)\})$$

for some one-chain $\{g_U\}$ with coefficients in $\mathscr{O}_X$ such that

$$s^* f_{U,V} = u^* f_{U,V} = s^*(g_U - g_V) \quad \text{and} \quad t^* f_{U,V} = v^* f_{U,V} = t^*(g_U - g_V)\,.$$

Let $\eta_U = \omega_U - dg_U$. Now

$$s^*\eta_U - s^*\eta_V = s^* df_{U,V} - s^* d(g_U - g_V) = 0$$

by the conditions that $\{g_U\}$ must satisfy and the fact that $(\{\omega_U\}, \{f_{U,V}\})$ is a hypercocycle. Similarly, $t^*\eta_U - t^*\eta_V = 0$. Let η_s and η_t be the the elements of Ω_S^1 determined by the cocycles $\{s^*\eta_U\}$ and $\{t^*\eta_U\}$ respectively.

Now to compute $\nabla h([\omega])$ we must lift $\bar{\omega}_U - d_{X/S}g_U$ to a section of $\Omega^1_{X,Z}$. Let $e_{s,U}$ and $e_{t,U}$ be elements of $\mathscr{O}_X(U)$ such that $s^* e_{s,U} = 1\, t^* e_{t,U} = 0$, $t^* e_{t,U} = 1$ and $s^* e_{t,U} = 0$. These elements exist since Z is étale over S. Then $\eta_U - (e_{s,U}\eta_s + e_{t,U}\eta_t)$ is such a lifting. To compute $\nabla h([\omega])$ we must take the hyper-coboundary of $(\{\eta_U - (e_{s,U}\eta_s + e_{t,U}\eta_t)\}, \{f_{U,V} - (g_U - g_V)\})$. It is

$$(\{\eta_s \otimes d_{X/S}e_{s,U} + \eta_t \otimes d_{X/S}e_{t,U}\}, \{\eta_s \otimes (e_{s,U} - e_{s,V}) + \eta_t \otimes (e_{t,U} - e_{t,V})\}, 0)\,.$$

The class of this hypercocycle is the image of

$$\eta_t - \eta_s \in \Omega_S^1 \quad \text{in} \quad \Omega_S^1 \otimes H^1_{DR}(X/S, Z)$$

(recall that we've determined a map of $K[S]$ into $H^1_{DR}(X/S, Z)$). Hence $\nabla h([\omega]) = \eta_t - \eta_s$.

The proposition now follows from the fact that

$$(\{\eta_s + ds^* g_U\}, \{s^* g_U - s^* g_V\}) = u^*(\{\omega_U\}, \{f_{U,V}\})$$

and

$$(\{\eta_t + dt^* g_U\}, \{t^* g_U - t^* g_V\}) = v^*(\{\omega_U\}, \{f_{U,V}\})\,. \qquad \square$$

COROLLARY 1.3.4. *If, in the above, u and v are constant, then* $M(s, t) = 0$.

4. ABELIAN SCHEMES

Suppose now that A is an Abelian scheme over S. Let $m: A \times_S A \to A$ be the addition law and e the zero section. For $s, t \in A(S)$, let $M(s) = M(e, s)$ and $s + t = m(s, t)$.

THEOREM 1.4.1. *The map M from $A(S)$ to* $\mathrm{Ext}(H^1_{DR}(A/S), K[S])$ *is a homomorphism.*

Proof. Let s and t be elements of $A(S)$. Define the map $g: A \to A$ by $g = m \circ (id, t \circ f)\,(g(x) = x + t(f(x)))$. Then $g^*: H^1_{DR}(A/S) \to H^1_{DR}(A/S)$ is the identity so that $g^*M(e, s) = M(e, s)$ on the one hand and $g^*M(e, s) = M(t, s + t)$ by Proposition 1.3.2 on the other. Hence,

$$M(s) + M(t) = M(e, s) + M(e, t) = M(t, s + t) + M(e, t) = M(e, s + t)$$

by Proposition 1.3.1. □

Let (B, τ) denote the $K(S)/K$ trace of $A_{K(S)}$ (see [L-AV]). In particular, B is an Abelian scheme over K and $\tau: B \times \operatorname{spec}(K(S)) \to A_{K(S)}$ is a homomorphism. Since K has characteristic zero τ is a closed immersion. Philosophically, B is the largest constant Abelian subscheme of $A_{K(S)}$ defined over over K. The morphism τ extends uniquely to an S-morphism $\bar{\tau}: B \times_K S \to A$. It follows that $B(K)$ maps naturally into $A(S)$. We call the elements s of $A(S)$ such that ns is in the image of $B(K)$, the constant sections of A/S.

PROPOSITION 1.4.2. *The kernel of M contains all constant sections of A/S.*

Proof. Let s be a constant section of A/S. Then there exists a positive integer n such that $ns = \bar{\tau} \circ (t \times id)$ where $t \in B(K)$. Hence it follows from the above theorem, Proposition 1.3.2 and Proposition 1.3.4 that $nM(s) = M(ns) = M(\bar{\tau}(t \times id)) = \bar{\tau}^*M(t \times id) = 0$. Since

$$\operatorname{Ext}(H^1_{DR}(A/S), K[S])$$

is uniquely divisible, by Corollary 1.1.2, the proposition follows. □

We wish to prove the conserve of this proposition. I.e. we wish to prove:

THEOREM 1.4.3. *The kernel of M is precisely the group of all constant sections of A/S.*

We will give two proofs of this result. The first is Algebraic. The second is analytic and is essentially a reformulation of Manin's proof based on remarks by Katz [K2] in a letter to Ogus.

5. THE ALGEBRAIC PROOF

a. *Differentials with logarithmic singularities*

(See [K] § 1.0). Suppose X is a smooth scheme over a scheme T and Z is a hypersurface in X whose irreducible components are smooth over T and