6. The analytic proof

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **36 (1990)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

that the image of $t-s$ in $J(C)$ is a constant section which completes the proof. $\square$

6. The Analytic Proof

In this section we will suppose $K = \mathbf{C}$.

a. *The Poincaré Lemma*

Suppose $(\mathscr{S}, \nabla)$ is a sheaf on S^{an} with integrable connection. Then by the Poincaré lemma for integrable connections, it follows that the complex of sheaves

$$\mathscr{S} \xrightarrow{\nabla} \Omega^1_{S^{an}} \otimes \mathscr{S} \xrightarrow{\nabla} \Omega^2_{S^{an}} \otimes \mathscr{S} \xrightarrow{\nabla} \dots$$

is a resolution of the sheaf $\mathscr{S}^{\nabla}$. Hence,

Proposition 1.6.1. *$H^i(\mathscr{S}, \nabla)$ is naturally isomorphic to $H^i(S, \mathscr{S}^{\nabla})$.*

Remark. As in Proposition 1.1.1, $H^1(\mathscr{S}, \nabla)$ is isomorphic to $\mathrm{Ext}(\mathscr{S}^{\vee}, \mathscr{O}_{S^{an}})$. We can describe the isomorphism from $H^1(\mathscr{S}, \nabla)$ to $H^1(S, \mathscr{S}^{\nabla})$ explicitly as follows: Let h be an element of $H^1(\mathscr{S}, \nabla)$. Let $\mathscr{C}$ is a covering of S by open disks. Suppose $\mathscr{E}$ is an extension of $\mathscr{S}^{\vee}$ by $\mathscr{O}_{S^{an}}$ corresponding to h. Then $\mathscr{E}^{\vee}$ is an extension of $\mathscr{O}_{S^{an}}$ by $\mathscr{S}$. For each $U \in \mathscr{C}$, there exists an $s_U \in \mathscr{E}^{\vee}(U)^{\nabla}$ which maps to 1 in $\mathscr{O}_{S^{an}}(U)$. Then the image h in $H^1(S, \mathscr{S}^{\nabla})$ is the class of the cocycle $\{(U, V) \to s_U - s_V\}$.

Suppose, X is a smooth proper S-scheme and Z is a subscheme of X which is either empty or finite over S. We will define the Betti homology sheaf $\mathscr{H}_1(X/S, Z, \mathbf{Z})$ on S^{an} as follows. If Z is smooth over S, we define $\mathscr{H}_i(X/S, Z, \mathbf{Z})$ to be the sheaf associated to the presheaf

$$U \to H_i(f^{-1}(U), f^{-1}(U) \cap Z, \mathbf{Z}),$$

(this latter group is the Betti homology of $f^{-1}(U)$ relative to $f^{-1}(U) \cap Z$). More generally, let S' be a non-empty affine open subset of S such that $Z' = Z \times_S S'$ is étale over S'. Let $X' = X \times_S S'$ and let ι denote the inclusion morphisms $X' \to X$, $Z' \to Z$ and $S' \to S$. We set

$$\mathscr{H}_i(X/S, Z, \mathbf{Z}) = \iota_* \mathscr{H}_i(X'/S', Z', \mathbf{Z}).$$

This is independent of the choice of S'. We also set

$$\mathscr{H}_i(X/S, \mathbf{Z}) = \mathscr{H}_i(X/S, \varnothing, \mathbf{Z}) \text{ and } \mathscr{H}_1(X/S, Z, \mathbf{C}) = \mathscr{H}_1(X/S, Z, \mathbf{Z}) \otimes \underline{\mathbf{C}}.$$

Suppose s and t are two distinct sections of X/S and $Z = s \cup t$. Suppose S' is an affine open of S such that Z' is étale ove S' in the notation of the previous paragraph. We have exact sequences

$$0 \to \mathscr{H}_1(X/S, \mathbf{Z}) \to \mathscr{H}_1(X/S, Z, \mathbf{Z}) \to \iota_* \mathscr{H}_0(Z'/S', \mathbf{Z}) \to \mathscr{H}_0(X/S, \mathbf{Z}) \to 0 .$$

and

$$0 \to \mathscr{H}_0(S'/S', \mathbf{Z}) \to \mathscr{H}_0(Z'/S', \mathbf{Z}) \to \mathscr{H}_0(X'/S', \mathbf{Z}) ,$$

where the first map is $t_* - s_*$. From which we derive the short exact sequence

$$0 \to \mathscr{H}_1(X/S, \mathbf{Z}) \to \mathscr{H}_1(X/S, Z, \mathbf{Z}) \to \underline{\mathbf{Z}} \to 0 .$$

since $\iota_* \underline{\mathbf{Z}}|_{S'^{an}} \cong \underline{\mathbf{Z}}$. In particular, if U is an open disk in S^{an}, we have an exact sequence

$$0 \to \mathscr{H}_1(X/S, \mathbf{Z})(U) \to \mathscr{H}_1(X/S, Z, \mathbf{Z})(U) \to \mathbf{Z} \to 0$$

We define the Betti cohomology sheaf $\mathscr{H}^1(X/S, Z, \mathbf{C})$ in the same way and it is easy to see that $\mathscr{H}^1(X/S, Z, \mathbf{C}) \cong \mathrm{Hom}(\mathscr{H}_1(X/S, Z, \mathbf{C}), \underline{\mathbf{C}})$. Also, it is known that if Z is étale over S then $\mathscr{H}^1(X/S, Z, \mathbf{C}) \cong R^1_{f_*}\mathscr{S}_Z$ where $\mathscr{S}_Z$ is the subsheaf of $\underline{\mathbf{C}}$ whose sections vanish on Z.

Suppose X is proper over S with connected fibers. Let

$$(\mathscr{H}^1_{DR}(X/S, Z), \nabla) = \mathscr{O}_{S^{an}} \otimes_{\mathscr{O}_S} (H^1_{DR}(X/S, Z), \nabla) .$$

We claim, for $Z \subseteq X$ finite over S.

$$(\mathscr{H}^1_{DR}(X/S, Z), \nabla) \cong (\mathscr{O}_{S^{an}} \otimes \mathscr{H}^1(X/S, Z, \mathbf{C}), d \otimes id)$$

This follows from the relative Poincaré lemma above on S' and hence on all of S since both sides are integrable connections. Hence,

LEMMA 1.6.2. *There is a natural isomorphism*

$$(\mathscr{H}^1_{DR}(X/S, Z)^{\vee}, \overset{\vee}{\nabla}) \cong (\mathscr{O}_{S^{an}} \otimes \mathscr{H}_1(X/S, Z, \mathbf{C}), d \otimes id) .$$

In particular

$$H^i(\mathscr{H}^1_{DR}(X/S, Z)^{\vee}, \overset{\vee}{\nabla}) \cong H^i(S^{an}, \mathscr{H}_1(X/S, Z, \mathbf{C})) .$$

We conclude, using this, Proposition 1.1.1 and GAGA that

THEOREM 1.6.3. *There exists a natural isomorphism*

$$\beta : \mathrm{Ext}(H^1_{DR}(X/S), \mathscr{O}_S) \to H^1(S^{an}, \mathscr{H}_1(X/S, \mathbf{C})) .$$

b. *End of Analytic Proof*

Now suppose X is an Abelian scheme over S. We have an exact sequence of sheaves over S^{an},

$$0 \to \mathscr{H}_1(X/S, \mathbf{Z}) \to \mathscr{L}ie_{X^{an}/S^{an}} \to \underline{X}^{an} \to 0 \ .$$

From the corresponding long exact sequence of cohomology groups we obtain an exact sequence

$$\mathscr{L}ie_{X^{an}/S^{an}}(S^{an}) \to X^{an}(S^{an}) \xrightarrow{\delta} H^1(S^{an}, \mathscr{H}_1(X/S, \mathbf{Z})) \ .$$

We may describe $\delta(s)$ as follows: Suppose $e \neq s$. Let $Z = e \cup s$. Then as $f_*(\Omega^1_{X^{an}/S^{an}})$ maps into $\mathscr{H}^1_{DR}(X/S)$, $\mathscr{H}_1(X/S, Z, \mathbf{Z})$ maps into

$$f_*(\Omega^1_{X^{an}/S^{an}})^{\vee} = \mathscr{L}ie_{X^{an}/S^{an}}$$

so that the diagram

$$\begin{tikzcd} \mathscr{H}_1(X/S, Z, \mathbf{Z}) \arrow[dr] & \\ \mathscr{H}_1(X/S, \mathbf{Z}) \arrow[u] \arrow[r] & \mathscr{L}ie_{X^{an}/S^{an}} \end{tikzcd}$$

commutes. Let $\mathscr{U}$ be an ordered covering of S by open disks. For each $U \in \mathscr{U}$ let $\gamma_U \in \mathscr{H}_1(X/S, Z, \mathbf{Z})(U)$ such that $\gamma_U \to 1$ under the map $\mathscr{H}_1(X/S, Z)(U) \to \mathbf{Z}$. Then the image of γ_U in $X(U)$ is $s(U)$. Hence $\delta(s)$ is represented by the one cocycle $\{(U, V) \to \gamma_U - \gamma_V\}$.

Now, it follows from this and the remark after Proposition 1.6.1 that $\beta \circ M$ is equal to the composition of δ and the natural map

$$H^1(S^{an}, \mathscr{H}_1(X/S, \mathbf{Z})) \to H^1(S^{an}, \mathscr{H}_1(X/S, \mathbf{C}) \cong H^1(S^{an}, \mathscr{H}_1(X/S, \mathbf{Z})) \otimes \mathbf{C} \ .$$

Hence, if $s \in X^{an}(S^{an}), M(s) = 0$ iff there exists a positive integer n such that $\delta(ns) = n\delta(s) = 0$. Hence ns is in the image of $\mathscr{L}ie_{X^{an}/S^{an}}(S^{an}) \to X^{an}(S^{an})$ and so is an infinitely divisible element of $X^{an}(S^{an})$.

Suppose $s \in X(S)$. We claim ns is an infinitely divisible element of $X(S)$. Let m be a positive integer. Let $t \in X^{an}(S^{an})$ such that $mt = ns$. There exists a finite étale Galois covering $\tilde{S}$ of S such that $t \in X(\tilde{S})$. If $\sigma \in \mathrm{Gal}(\tilde{S}/S)$, then $t^{\sigma} = t$ because $t^{\sigma}(x) = t(\sigma^{-1}(x))$ for $x \in \tilde{S}(\mathbf{C})$. It follows that $t \in X(S)$. This establishes our claim.

Finally, it follows from the function field Mordell-Weil Theorem [LN] that the image of ns in $X_{\mathbf{C}(S)}(\mathbf{C}(S))$ is a constant section X/S. Theorem 1.4.3 now follows immediately. $\square$