# III. MORDELL'S CONJECTURE

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **36 (1990)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

## III. Mordell's conjecture

Suppose $L$ is a field of characteristic zero of finite type over a relatively algebraically closed subfield $K$.

THEOREM 3.1 (Manin). *Suppose $C$ is a curve of genus at least 2 defined over $K$. Suppose $C(L)$ is infinite, then there exists a curve $C_0$ defined over $K$ such that $C_0 \times_K L \cong C$ and $C(K)$ minus the image of $C_0(K)$ under this isomorphism is finite.*

We can translate this into

THEOREM 3.1 (BIS). *Suppose $S$ is a variety defined over $K$ and suppose $C \to S$ is a smooth proper curve of genus at least 2 over $S$. Suppose $C(S)$ is infinite, then there exists a curve $C_0$ defined over $K$ such that $C_0 \times_K S \cong C$ and $C(S)$ minus the image of $C_0(K)$ under this isomorphism is finite.*

*Remarks.* First, it is possible to reduce this by standard arguments to the case in which $S$ is a smooth affine curve over $K$ and so we will suppose this to be the case. Second, if we can prove that $C_0 \times_K X \cong C$ for some $C_0$ defined over $K$, (i.e. that $C$ is a constant family) then this is de Franchis' theorem which is proven in Lang's Fundamentals of Diophantine Geometry. Hence to prove this theorem all we have to do is show that if $C(S)$ is infinite then $C$ is a constant family of curves.

### 1. Sets of bounded height

In this section we will either recall or derive the properties of heights needed in the sequel.

Let $f: X \to S$ be a smooth projective morphism of varieties over $K$ a field of characteristic zero. Corresponding to a projective embedding of $X$ over $S$, there exists a function $h: X(S) \to \mathbf{R}$ called a logarithmic height. (For a reference, see ([L-FD] Chapter 3, §3). If the logarithmic height of a subset of $X(S)$ is bounded with respect to one projective embedding, it is bounded with respect to all (See [L] Prop. 1.7, Chapt. 4). We will call such a set a set of bounded height and a set of points which is not of bounded height, a set of unbounded height. We will need several properties of such sets. If $g: X' \to X$ is a morphism of projective schemes over $S$ which is finite onto its image, then the inverse image of a set of bounded height in $X(S)$ is a set of bounded height in $X'(S)$. Suppose $X$ is an Abelian scheme over $S$ and $R$ is the subgroup of $X(S)$ consisting of constant sections of $X/S$. Let $s \in X(S)$. Then the set $s + R$ is a set of bounded height.

LEMMA 3.1.1 (Manin). *Suppose $E$ is a finite dimensional $K$ vector subspace of $K(C)$. Then the set*

$$T = \{s \in C(S) : \exists k \neq 0 \in E \quad \text{such that} \quad s^*k = 0\}$$

*has bounded height.*

*Proof.* Without loss of generality we may increase $E$ to suppose that the rational map $g: C \to \mathbf{P}_K(E)$ given on points by $x \to (e \in E \to e(x))$ is birational onto its image (note: $g$ is actually a morphism on the compliment of the polar locus of $E$). It follows that $g$ induces an embedding of the generic fiber of $C/S$ into $\mathbf{P}_{K(S)}(E \otimes K(S))$. Let $h$ denote the logarithmic height with respect to this embedding. It follows that if $s \in C(S)$, $g \circ s$ is constant or $g \circ s$ has degree one. In the former case $h(s)$ is zero and the degree of the Zariski closure of $g \circ s(S)$ in $\mathbf{P}(E)$ in the latter.

Now if $s \in T$, and $g \circ s$ is not constant, it follows that the Zariski closure of $g \circ s(S)$ is a component of a hyperplane section of the Zariski closure of $g(C)$. Hence, $h(s)$ is less than or equal to the degree of the Zariski closure of $g(C)$. This proves the lemma. $\square$

The key property about heights we will need is:

THEOREM 3.1.2. *Suppose $C \to S$ is as in the above theorem. If $C(S)$ contains an infinite set of bounded height then $C$ is a constant family.*

(See Corollary 2.2, Chapter 8 of [L-FD].)

Hence all we need prove is that the elements of $C(S)$ have bounded height.

## 2. LANG-SIEGEL TOWERS

Suppose the genus of $C$ is at least 1. Suppose $T$ is an infinite subset of $C(S)$.

PROPOSITION 3.2.1. *There exists a projective system of curves*

*$(\{C_n\}, \{h_{m,n}\})$, $m, n \in \mathbf{Z}_{>0}$ and $n \leqslant m$, over $K$ such that*

(i) $C_1 = C$,

(ii) *$h_{m,n}: C_m \to C_n$ is étale,*

(iii) $(h_{m,1})^{-1}(T) \cap C_m(S)$ *is infinite,*

(iv) *There exists a finite covering* $S_{m,n}$ *of* $S$ *such that the fiber product of* $h_{m,n}$ *with* $S_{m,n}$ *is Galois, Abelian and of positive degree.*

Let $J$ denote the Jacobian scheme of $C$ over $S$. Let $a: C \to J$ be an Albanese morphism. Let $p$ be a prime. Let $\bar{T}$ denote the closure of $a(T)$ in $J(S) \otimes \mathbf{Z}_p$. Since $a(T)$ is infinite it follow from the Mordell-Weil Theorem that there exists a $t \in \bar{T} - a(T)$. Let $t_n \in T$ such that $t - a(t_n) \in p^n J(S)$. Let $C_n$ denote the normalization of the fiber-product of $C$ and $J$ via the map $H_n: x \to p^n x + t_n$ and $h_{n,1}$ the natural map from $C_n$ to $C$. It follows that $C_n$ is defined over $S$ and since $H_m(J(S)) \supseteq \{t_n : m \mid n\}$ that $h_{n,1}(C_m(S))$ contains an infinite subset of $T$.

All that remains is to exhibit the maps $h_{m,n}$. Clearly, $t_m - t_n = p^n r_{m,n}$ for some $r_{m,n} \in J(S)$. Let $H_{m,n}$ denote the map $x: p^{m-n}x + r_{m,n}$. Then $H_{m,k} = H_{n,k} \circ H_{m,n}$. It follows that $H_{m,n}$ pulls back to a morphism $h_{m,n}: C_m \to C_n$. It is easy to see that this morphism becomes Abelian after adjoining the $p^{m-n}$-torsion points on $J$. This proves the proposition. □

*Remark.* One can also prove the above proposition with the condition $n \leqslant m$ replaced by $n \mid m$.

## 3. Corollaries of the Theorem of the Kernel

Lemma 3.3.1. *Suppose* $g: X' \to X$ *is a morphism of smooth proper schemes with geometrically connected fibers over* $S$. *Then if* $\mu \in PF(X'/S)$ *and* $s, t \in X(S)$, $(g^*\mu)(s, t) = \mu(g \circ s, g \circ t)$.

*Proof.* This follows easily from Lemma 1.3.2. □

Suppose $J$ is the Jacobian of $C$ over $S$ and $g$ is an Albanese morphism, then since $g^*: H^1_{DR}(J/S) \to H^1_{DR}(C/S)$ is an isomorphism $g^*: PF(J/S) \to PF(C/S)$ is an isomorphism.

Lemma 3.3.2. *Let* $\mu$ *be a fixed Picard-Fuchs differential equation on* $C/S$. *Then* $\{\mu(s, t): s, t \in C(S)\}$ *lies in a finite dimensional subspace of* $K[S]$ *over* $K$.

*Proof.* Suppose $\tilde{\mu} \in PF(J/S)$ such that $g^*\tilde{\mu} = \mu$. The lemma follows from the Mordell-Weil theorem which together with the Theorem of the kernel implies that $J(S)$ modulo the kernel of the homomorphism $s \to \tilde{\mu}(e, s)$ is a finitely generated Abelian group. □

LEMMA 3.3.3. *Suppose $A$ is an Abelian scheme over $S$ such that $[W_{A/S}] = H^1_{DR}(A/S)$ and $g: C \to A$ is a non-constant morphism over $S$. Fix $s \in C(S)$. Then the set $T = \{t \in C(S): (g^*\mu)(s, t) = 0$ for all $\mu \in PF(A/S)\}$ is of bounded height.*

*Proof.* Let $A'$ denote the smallest Abelian subscheme of $A$ over $S$ containing $g(C)$. Since the map $g^*: PF(A/S) \to PF(A'/S)$ is surjective and $[W_{A/S}] = H^1_{DR}(A/S)$, it follows from Proposition 2.1.2 that $g(T)$ is contained in a translation of the group of constant sections of $A'/S$. Hence, $g(T)$ is a set of bounded heigt. Finally, since $C \to g(C)$ is a finite morphism, it follows that $T$ is a set of bounded height. □

In particular,

COROLLARY 3.3.4. *Suppose $A$ is an Abelian scheme over $S$ such that $\kappa_{A/S}$ is an isomorphism and $g: C \to A$ is a non-constant morphism over $S$. Fix $s \in C(S)$. Then the set $\{t \in C(S): (g^*\mu_\omega)(s, t) = 0$ for all $\omega \in \omega_{A/S}\}$ is of bounded height.*

## 4. PROOF OF MORDELL'S CONJECTURE

PROPOSITION 3.4.1. *Suppose the kernel of the $\kappa_{C/S}$ has rank at least 2 over $K[S]$, then the points of $C(S)$ have bounded height.*

*Proof.* Suppose $C(S)$ contains points of arbitrarily large height. Fix $s \in C(S)$. By shrinking $S$, if necessary, we may suppose that there exists a function $z \in K[S]$ such that $\Omega^1_S = K[S]dz$ and there exists a finite covering $\mathscr{L}$ of $C$ by affine opens $U$ and functions $v_U \in \mathscr{O}_C(U)$ such that $s \in U(S)$, and $\Omega^1_C(U)$ is spanned by $dz$ and $dv_U$. We may also suppose that $s^*v_U = 0$ by replacing $v_U$ with $v_U - (s \circ f)^* v_U$ if necessary. For $U \in \mathscr{L}, u \in \mathscr{O}_C(U)$ we define $\partial_{U,z}u$ and $\partial_{U,v}u$ by the equation

$$du = \partial_{U,z}u dz + \partial_{U,v}u dv_U .$$

Then $\partial_{U,z}$ is a lifting of $\partial =: \partial/\partial z$. We set $\mu(t) = \mu(s, t)$ for

$$\mu \in PF =: PF(C/S)$$

and $t \in C(S)$.

Let $\omega_1$ and $\omega_2$ be two independent elements in the kernel of $\kappa_{C/S}$. It follows that there exist $\omega_1'$ and $\omega_2' \in \omega_{C/S}$ such that

$$\partial[\omega_i'] = [\omega_i'] .$$

Hence $\mu_i = \partial \otimes \omega_i - 1 \otimes \omega_i'$ is in $PF$. For $U \in \mathscr{U}$ let $w_{U,i}$ and $u_{U,i}$ be elements of $\mathscr{O}_C(U)$ such that

$$\partial_{U,z}\omega_i - \omega_i' = d_{C/S} w_{U,i} ,$$

$s^* w_{U,i}$ and $\omega_i = u_{U,i} d_{C/S} v_U$ on $U$. Let $T$ denote the set of $t \in C(S)$ such that $t \cap U \neq \varnothing$ and $t^* v_U \neq 0$ for all $U$ *in* $\mathscr{U}$. This is the complement of a finite subset. For $t \in T$

$$\mu_i(t) = t^*(w_{U,i}) + t^*(u_{U,i})\partial t^*(v_U) \tag{4.1}$$

for all $U \in \mathscr{U}$, by Corollary 2.2.4.

For $t \in T$, $U \in \mathscr{U}$ let

$$h_{U,t} = u_{U,2}\mu_1(t) - u_{U,1}\mu_2(t) - (u_{U,2}w_{U,1} - u_{U,1}w_{U,2}) .$$

We deduce from (4.1) that $t^* h_{U,t} = 0$. On the other hand, by Lemma 3.3.2, the set of functions $h_{U,t}$ lies in a subspace of $\mathscr{O}_C(U)$ of finite dimension over $K$. It follows from Lemma 3.1.1 that $h_{U,t} = 0$ for all $t$ in in a subset $T'$ of $T$ of unbounded height. Fix $t_0 \in T'$, and set $c_i =: \mu_i(t_0)$, then it follows that

$$u_{U,2}(\mu_1(t) - c_1) - u_{U,1}(\mu_2(t) - c_2) = 0$$

for all $t \in T'$. Now since $\omega_1$ and $\omega_2$ are independent over $K[S]$, $u_{U,1}$ and $u_{U,2}$ are independent over $K(S)$ and so we must have

$$\mu_i(t) = c_i$$

for all $t \in T'$. Let $z_{U,i} = u_{U,i}^{-1}(c_i - w_{U,i})$. Let $z_{U,i} = u_{U,i}^{-1}(c_i - w_{U,i})$. Let $T''$ denote the subset of $T'$ such that $t^* u_{U,1} \neq 0$ and $t^* u_{U,2} \neq 0$ for all $U \in \mathscr{U}$. This is the complement of a finite subset of $T$. For $t \in T'$

$$t^* z_{U,i} = \partial t^* v_U \tag{4.2}$$

for all $U \in \mathscr{U}$. This implies that $z_{U,1} = z_{U,2}$ since $T''$ is infinite. Set $z_U = z_{U,1}$.

Set $u_U = u_{U,1}$ and $w_U = w_{U,1}$. On $U \cap V$,

$$dv_V = g_{U,V} dz + f_{U,V} dv_U$$

for some $g_{U,V} \in \mathscr{O}_C(U)$ and $f_{U,V} \in \mathscr{O}_C(U \cap V)^*$. It follows that

$$u_U = f_{U,V} u_V, \qquad \partial_{U,v} g_{U,V} = \partial_{V,z} f_{U,V} \qquad \text{and} \qquad w_U = w_V + u_V g_{U,V} .$$

Hence

$$z_U = f_{U,V} z_v - g_{U,V} .$$

Hence, we may define a divisor $Y$ which on $U$ is the polar divisor of $z_U$. (It is clear that the support of $Y$ is contained in the intersection of the supports of the divisors of $\omega_1$ and $\omega_2$.) Let $C' = C - Y, U' = U \cap C'$ for $U \in \mathscr{L}, \upsilon_{U'} = \upsilon_U|_{U'}$ etc. Then the above implies that we may define a lifting $\tilde{\partial}$ of $\partial$ to $\Gamma(\mathscr{D}er_{C'/K})$ such that on $U'$,

$$\tilde{\partial}\upsilon_{U'} = z_{U'} \ .$$

If $Y = \varnothing$, this implies that $\kappa_{C/S}$ is zero and hence that $C/S$ is isoconstant. This contradicts de Franchis' theorem. Thus $Y = \varnothing$.

It follows from (4.2) that $t \cap Y = \varnothing$ for all $t \in T''$. In particular, $Y$ has no vertical components. But this contradicts the function field analogue of Siegel's theorem [L-IP] since $T''$ is a set of unbounded height. This completes the proof of the proposition. □

*Remark.* In the appendix we will present Manin's original proof of this proposition which uses Theorem 2.1.0 and does not use Siegel's theorem. To this end, we point out that it follows from (4.2) that

$$t^*\tilde{\partial}x = \partial(t^*x) \tag{4.3}$$

for all $x \in K[C']$ and $t \in T''$.

We will now complete the proof of the function field Mordell conjecture. The argument here is essentially the same as that in Manin's paper except that we found it necessary to be more careful about the choice of base points. Suppose $C/S$ is a curve over $S$ such that $C(S)$ contains points of arbitrarily large height. Let $(\{C_n\}, \{h_{m,n}\})$ be the projective system as described in § 3.2 such that $C_1 = C$ and $C_n(S)$ contains points of arbitrarily large height. From the previous proposition, we know that the rank of the kernel of the $\kappa_{C_n/S}$ is at most one. Since these ranks grow with $n$, by replacing $C$ with $C_n$ for appropriate $n$, we may suppose these ranks are all equal. Set $h_m = h_{m,1}$ .

By shrinking $S$, we may suppose that there exists a $z \in K[S]$ such that $dz$ spans $\Omega^1_S$ over $k[S]$. Let $\partial = \partial/\partial z$.

Let $J_n$ denote the Jacobian of $C_n$ and $A_n = J_n/h_n^* J_1$. It follows that $\kappa_{A_n/S}$ is an isomorphism. We identify $\omega_{A_n/S}$ with its image via an Albanese pullback in $\omega_{C_n/S}$. Recall that in these circumstances we have a Picard-Fuchs equation $\mu_\omega = : \mu_{\partial,\omega}$ attached to $\omega \in \omega_{A_n/S}$.

Fix an $s \in C(S)$. By shrinking $S$ if necessary, we may suppose there is an affine open $U$ of $C$ such that $s \in U(S)$ and there exists an element $\upsilon$ of $\mathscr{O}_C(U)$ such that $\Omega^1_{C/S}(U)$ is spanned by $d_{C/S}\upsilon$ over $\mathscr{O}_C(U)$ and, $s^*\upsilon = 0$. Recall, for $u \in \mathscr{O}_C(U)$ we defined $\partial_z u$ and $\partial_\nu u$ by

$$du = \partial_z u dz + \partial_v u dv .$$

Now suppose $n \geqslant 1$ and $S'$ is an étale (not necessarily finite) connected open of $S$ such that $C_n'(S')$ contains a point $r$ lying overs $s$. Let $C_n' = C_n \times S'$ and $A_n' = A_n \times S'$. We will abuse notation for the moment and let $z$ and $v$ denote their pullbacks to $S'$ and $C_n'$ respectively. Let $h_n' \colon C_n' \to C_1'$ denote the pullback of $h_n$. Let $U'$ denote the inverse image of $U$ in $C_1'$. We set $U_n = h_n'^{-1}(U')$. Then since $h_n'$ is unramified, $dz$ and $dv$ span $\Omega^1_{C_n'}(U_n)$. In these circumstances we have a $K$-linear map $L_{z,v,r} \colon \omega_{A_n'/S'} \to K(C_n)^4$ described in Corollaries 2.2.4 and 2.2.5.

Let $n, S', r$ be such that the dimension of the $K(C_n)$-span of the image of $L_{z,v,r}$ is maximal over all such triples. Call that dimension $R$. Now fix $m > n$ and replace $S$ with an étale open of $S'$ such that, $C_m$ is Galois over $C_n$ with Galois group $G$ and there exists an $r' \in C_m(S)$ above $r$. Let $w = h_n^* v, h = h_{m,n}$ and let $Y = C_n$ and $X = C_m$. Our hypotheses imply, in particular, that $X(S)$ is of unbounded height. Let $B = J_m / h^* J_n$. Then, $\kappa_{B/S}$ is an isomorphism. The module, $\omega_{B/S}$ injects naturally into $\omega_{X/S}$ and we identify it with its image.

Let $\eta_1, \dots, \eta_n$ be a $K(S)$-basis for $\omega_{B/S}$. Let $L = L_{z,h^*w,r}$. As $L \circ h^* = L_{z,w,r}$ our maximality hypothesis implies that $L(h^*\omega_{A_n/S}) \subseteq L(\omega_{A_n'/S})K(X)$ and so there exist elements $\omega_1, \dots, \omega_R \in \omega_{A_n/S}$ and elements $z_{ij} \in K(X)$ such that

$$L(\eta_i) = \sum z_{i,j} L(h^*\omega_j) .$$

Let

$$T = \{t \in X(S) \colon t \cap U_m \neq \varnothing, t^* w \neq 0\} .$$

The complement of $T$ in $X(S)$ is finite. In particular, in the notation of Corollary 2.2.5, since $V_{z,h^*w}(t) = V_{z,W}(h(t)), t \in T$ and $L(h^*\omega) = L_{z,w,r}(\omega)$ for $\omega \in \omega_{A_n/S}$, by Corollary 2.2.5

$$\mu_{\eta_i}(r',t) = \sum t^* z_{i,j} \mu_{h^*\omega_i}(r',t) = \sum t^* z_{i,j} \mu_{\omega_j}(r, h(t)) .$$

for $t \in T$. Let

$$f_{i,t} = \mu_{\eta_i}(r',t) - \sum z_{i,j} \mu_{\omega_j}(r, h(t)) .$$

We see that $t^* f_{i,t} = 0$ and Lemma 3.3.2 implies that the set

$$\{f_{i,t} \colon 0 \leqslant i \leqslant k, t \in T\}$$

is contained in a finite dimensional $K$ subspace of $K(X)$. Hence by

Lemma 3.1.1, using the fact that height is stable under the action of $G$, the subset $T_1$ of $T$ consisting of elements $t$ for which there exists a $\sigma \in G$ and an $i$, $0 \leqslant i \leqslant n$, such that $f_{i,t^\sigma} \neq 0$ is of bounded height.

Let $T_2 = T - T_1$. Clearly, $T_2$ is stable under $G$. Moreover, $f_{i,t} = 0$ for all $t \in T_2$. That is,

$$\mu_{\eta_i}(r', t) = \sum z_{i,j} \mu_{\omega_j}(r, h(t)) .$$

In particular, $\mu_{\eta_i}(r', t^\sigma) = \mu_{\eta_i}(r', t)$ for $t \in T_2$ and $\sigma \in G$. On the other hand,

$$\mu_{\eta_i}(r', t^\sigma) = \mu_{\eta_i}(r', r'^\sigma) + \mu_{\eta_i}(r'^\sigma, t^\sigma) = \mu_{\eta_i}(r', r'^\sigma) + \mu_{\eta_i}\sigma(r', t)$$

by (II, 1.1) and Lemma 3.3.1. It follows that

$$\mu_{\omega - \omega^\sigma}(r', t) = \mu_\omega(r', r'^\sigma)$$

for all $\omega \in \omega_{B/S}$, $\sigma \in \mathrm{Gal}(X/Y)$ and $t \in T_2$. Let $t_0 \in T_2$. By (II, 1.1) we conclude that $\mu_{\omega - \omega^\sigma}(t_0, t) = 0$ for all $\omega \in \omega_{B/S}$, $\sigma \in \mathrm{Gal}(X/Y)$ and $t \in T_2$. But $\{\omega - \omega^\sigma : \omega \in \omega_{B/S},\ \sigma \in \mathrm{Gal}(X/Y)\}$ spans $\omega_{B/S}$ over $K$ by the definition of $B$. Corollary 3.3.4, applied to the morphism $X \to B$, implies $T_2$ is a set of bounded height. But this implies that $X(S)$ is a set of points of bounded height. This contradiction completes the proof of Mordell's conjecture for function fields. $\square$

## Appendix: Chai's proof of the Theorem of the Kernel

In this appendix, we give Chai's proof of Manin's Theorem of the Kernel, Theorem 2.1.0 above and explain how Manin used it to prove the function field Mordell conjecture. Let notation be as in Section II. As explained in that Section, the theorem follows from the assertion:

(A1) $N(e, s) = 0$ iff $w \circ N(e, s) = 0$.

Let $H = H^1_{DR}(A/S)$. For a subconnection $D$ of $H$, let $\tilde{D}$ denote the pullback of $H^1_{DR}(A/S, Z)$ to $D$. As (A1) is stable under fiber products and isogenies (see Proposition 1.3.2), (A1) is a consequence of the following theorem, taking $D = [W]$.

PROPOSITION A1.1. (Chai). *Suppose $A/S$ is irreducible and not isotrivial. Let $D$ be a non-trivial subconnection of $H$. Then the extension $\tilde{H}$ of $H$ of connections splits iff the extension $\tilde{D}$ of $D$ does.*