Appendix: Chai's proof of the Theorem of the Kernel

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Lemma 3.1.1, using the fact that height is stable under the action of G, the subset T_1 of T consisting of elements t for which there exists a $\sigma \in G$ and an i, $0 \leqslant i \leqslant n$, such that $f_{i,t^\sigma} \neq 0$ is of bounded height.

Let $T_2 = T - T_1$. Clearly, T_2 is stable under G. Moreover, $f_{i,t} = 0$ for all $t \in T_2$. That is,

$$\mu_{\eta_i}(r', t) = \sum z_{i,j} \mu_{\omega_j}(r, h(t)) .$$

In particular, $\mu_{\eta_i}(r', t^\sigma) = \mu_{\eta_i}(r', t)$ for $t \in T_2$ and $\sigma \in G$. On the other hand,

$$\mu_{\eta_i}(r', t^\sigma) = \mu_{\eta_i}(r', r'^\sigma) + \mu_{\eta_i}(r'^\sigma, t^\sigma) = \mu_{\eta_i}(r', r'^\sigma) + \mu_{\eta_i}\sigma(r', t)$$

by (II, 1.1) and Lemma 3.3.1. It follows that

$$\mu_{\omega - \omega^\sigma}(r', t) = \mu_\omega(r', r'^\sigma)$$

for all $\omega \in \omega_{B/S}$, $\sigma \in \mathrm{Gal}(X/Y)$ and $t \in T_2$. Let $t_0 \in T_2$. By (II, 1.1) we conclude that $\mu_{\omega - \omega^\sigma}(t_0, t) = 0$ for all $\omega \in \omega_{B/S}$, $\sigma \in \mathrm{Gal}(X/Y)$ and $t \in T_2$. But $\{\omega - \omega^\sigma : \omega \in \omega_{B/S},\ \sigma \in \mathrm{Gal}(X/Y)\}$ spans $\omega_{B/S}$ over K by the definition of B. Corollary 3.3.4, applied to the morphism $X \to B$, implies T_2 is a set of bounded height. But this implies that $X(S)$ is a set of points of bounded height. This contradiction completes the proof of Mordell's conjecture for function fields. $\square$

APPENDIX: CHAI'S PROOF OF THE THEOREM OF THE KERNEL

In this appendix, we give Chai's proof of Manin's Theorem of the Kernel, Theorem 2.1.0 above and explain how Manin used it to prove the function field Mordell conjecture. Let notation be as in Section II. As explained in that Section, the theorem follows from the assertion:

(A1) $N(e, s) = 0$ iff $w \circ N(e, s) = 0$.

Let $H = H^1_{DR}(A/S)$. For a subconnection D of H, let $\tilde{D}$ denote the pullback of $H^1_{DR}(A/S, Z)$ to D. As (A1) is stable under fiber products and isogenies (see Proposition 1.3.2), (A1) is a consequence of the following theorem, taking $D = [W]$.

PROPOSITION A1.1. (Chai). *Suppose A/S is irreducible and not isotrivial. Let D be a non-trivial subconnection of H. Then the extension $\tilde{H}$ of H of connections splits iff the extension $\tilde{D}$ of D does.*

Proof. The only if direction is clear.

For the other direction, we may, without loss of generality, suppose $K = \mathbf{C}$. Fix $q \in S(\mathbf{C})$. For an integrable connection D on S. Let D_q denote the fiber of D at q and $G(D)$ denote the Zariski closure of the image of the monodromy group at q of D in $\mathrm{End}_{\mathbf{C}}(D_q)$.

Let $N(D)$ denote the kernel of the natural map from $G(\tilde{D})$ to $G(D)$. The group $G(D)$ acts naturally by conjugation on $N(D)$. Moreover, since $\tilde{H}$ is an extension of H by $\mathbf{C}$, (recall (e, s) determines a basis) on which $G(\tilde{H})$ acts trivially and we have a natural $G(D)$-equivariant pairing $(\ ,\ ): N(D) \times D \to \mathbf{C}$ given by $(n, d) = n(d) - d$. Hence we have a commutative diagram

$$\text{(A2)} \qquad \begin{array}{ccc} N(H) & \to & H^* \\ \downarrow & & \downarrow \\ N(D) & \to & D^* \end{array}$$

where the right arrow is the natural surjection.

Now suppose that the extension $\tilde{D}$ of D splits. Then $N(D) = 0$ since $G(D)$ acts trivially on $\mathbf{C}$. By the Poincaré lemma the map of $\tilde{G}(H)$ modules $\tilde{H}_q \to H_q$ is defined over $\mathbf{Q}$. It follows from [D-H; Corollaire 4.4.15] that H_q^* is an irreducible representation. Hence $N(H) = 0$ or $N(H)$ surjects onto H_q^*. In the latter case, it follows from (A2) that $N(D)$ surjects onto D_q^* but this implies $D_q^* = (0)$, a contradiction. Thus $N(H) = 0$. This implies $G(H)$ acts on the exact sequence,

$$0 \to \mathbf{C} \to \tilde{H}_q \to H_q \to 0 \,.$$

As $G(H)$ is semi-simple by [D-H; Corollaire 4.2.9] we see that this sequence splits as well. This implies that the horizontal sequence

$$0 \to \mathcal{O}_S \to \tilde{H} \to H \to 0$$

splits by [D-SR; Proposition 1.3, Theorem 2.23 and Theorem 5.9]. $\square$

By replacing Proposition 2.1.2 by Theorem 2.1.0 in the proof of Lemma 3.3.3 one obtains:

COROLLARY A1.2. *The conclusions of Proposition 2.1.2 and Lemma 3.3.3 are true without the assumption that* $[W_{A/S}] = H^1_{DR}(A/S)$.

Now we give Manin's proof of Proposition 3.4.1 using Theorem 2.1.0. This was the only place in [*M*], where this theorem was needed. This proof does not use Siegel's Theorem.

Let notation be as in Section 3.4. Siegel's theorem was not used until the last paragraph of the proof of Proposition 3.4.1. Therefore we may assume C' is affine, $W(S)$ contains a set T' of unbounded height and we have a derivation $\tilde{\partial}$ on W such that $t^*\tilde{\partial}x = \partial(t^*x)$ for all $x \in K[W]$.

It follows from Lemma 2.2.3 that for each $\mu \in PF$, there exists an $x_\mu \in K[S]$ such that

$$\mu(t) = t^*x_\mu .$$

Lemma 3.3.2 implies that $\{\mu(t) - x_\mu : t \in T''\}$ is contained in a finite dimensional K-linear subspace of $K(C)$. Hence, by Lemma 3.1.1, $\mu(t) = x_\mu$ for all $\mu \in PF$ and all t in the complement T''' of a finite subset of T''. (We use here that PF is finitely generated over $\mathscr{D}_S$.) Fix $t_0 \in T'''$. Then $\mu(t_0, t) = 0$ for all $t \in T''$ and all $\mu \in PF$. This contradicts the above corollary and thus proves Proposition 3.4.1.