4. A LONG-STANDING CONJECTURE !

THEOREM 3.5 (S. A. Williams). *Let* Ω *be a bounded open subset such that the equation* $\Delta T + \alpha^2 T = -1_\Omega$ *has a function solution of compact support for some* $\alpha > 0$. *Let* R, K, L *be positive real numbers such that* $L > KR$. *Assume that for* $P \in \partial^* \Omega$ *there exists a coordinate system* (x, y) *around* P *so that*

(i) $Q = (-R, R) \times (-L, L)$ *intersects* $\partial\Omega$ *in the graph* $y = f(x)$ *of a Lipschitz function* f *with Lipschitz constant* K, *and*

(ii) $Q \cap \Omega = \{(x, y) : |x| < R$ *and* $f(x) < y < L\}$.

Then f *is real analytic in a neighbourhood of* P.

Thus if we restrict ourselves to the class $\mathscr{D}$ of simply connected bounded open sets with Lipschitz boundary then $\Omega \in \mathscr{D}$ can fail to have the Pompeiu property only if $\partial\Omega$ is real analytic.

4. A LONG-STANDING CONJECTURE!

The following Conjecture has received quite some attention in the literature ([3], [10], [34]).

Conjecture. If $\Omega \subseteq \mathbf{R}^2$ is in the class $\mathscr{D}$ described above and if Ω does not have the Pompeiu property, then Ω is a disc.

As pointed out before, the work of Williams shows that is is enough to consider Ω with $\partial\Omega$ real analytic. For $\Omega \in \mathscr{D}$, the existence of (a necessarily positive) α^2 for which (3.1) has a distribution solution of compact support is equivalent to the existence of a positive γ for which the following overdetermined system has a solution.

$$\text{(4.1)} \qquad \Delta T + \gamma T = 0 \quad \text{on} \quad \Omega$$

$$T = \text{constant} \neq 0 \quad \text{on} \quad \partial\Omega, \partial T/\partial n \equiv 0 \quad \text{on} \quad \partial\Omega$$

(see [34] for details). Thus the conjecture can be stated as follows:

If for $\Omega \in \mathscr{D}$, there exists $\gamma > 0$ for which (4.1) admits a solution, then Ω is a disc.

It is remarked in [34] that the conjecture is closely related to a result of Serrin ([25]): If Ω is a bounded connected open set with smooth boundary on which

$$\Delta u = -1 \quad \text{on} \quad \Omega$$

$$u = 0, \partial u/\partial n = \text{constant} \quad \text{on} \quad \partial\Omega$$

has a function solution, then Ω must be a disc.

We now state two partial answers to the conjecture that seem to support the conjecture.

THEOREM 4.1 (Berenstein [3]). *Let* Ω *be a simply connected bounded open subset of* $\mathbf{R}^2$ *with* $C^{2+\varepsilon}$ *boundary, where* $\varepsilon > 0$. *Assume that the boundary value problem (4.1) has solutions for infinitely many positive* γ, *then* Ω *is a disc.*

We need some notation for the next result due to Brown and Kahane ([10]). Let Ω be a convex bounded open connected subset of $\mathbf{R}^2$. For $0 \leqslant \theta < \pi$, let $\omega(\theta)$ be the distance between the two parallel support lines for Ω which make an angle θ with the positive real axis. We assume ∂D is smooth so that ω is a continuous function. Let

$$m(\Omega) = \inf\{\omega(\theta) : 0 \leqslant \theta < \pi\} \quad \text{and} \quad M(\Omega) = \sup\{\omega(\theta) : 0 \leqslant \theta < \pi\}.$$

THEOREM 4.2 (Brown and Kahane [10]). *Let* Ω *be a convex region of* $\mathbf{R}^2$ *with* $\partial\Omega$ *real analytic. If* $m(\Omega) \leqslant \frac{1}{2} M(\Omega)$, *then* Ω *has the Pompeiu property.*

We remark that the proof of this Theorem is elementary and very elegant.

5. POMPEIU PROPERTY IN NON-COMPACT SYMMETRIC SPACES

Let G *be a connected non-compact semisimple Lie group having finite centre and real rank 1.* Let K be a fixed maximal compact subgroup of G. The space G/K is then a globally symmetric space of the non-compact type of rank 1. G/K is equipped with a natural Riemannian structure with respect to which G acts as a group of isometries and the associated Riemannian volume element μ is G-invariant. The basic results for the Pompeiu problem in this set-up are due to Berenstein and Zalcman ([9], [4]) and Berenstein and Shahshahani ([7]). In [9], the Fourier-analytic characterisation of a set — in fact, more generally, a collection of sets — having the pompeiu property is obtained and some explicit computations are made for geodesic spheres. In [7], the Pompeiu problem is reduced to an eigenvalue problem as in Section 4 and the analogue of Williams's results is obtained. We shall mainly present here a result implicit in the work of Berenstein and Zalcman as well as Berenstein and Shahshahani from our