2.1. The Goeritz matrix and graph of a link

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Now if we adopt the usual convention (see [5]) that the value of $\sigma_L(\omega)$ at a root of the Alexander polynomial is defined to be the mean of its two "adjacent" values

$$\lim_{\varepsilon \to 0+} \sigma_L(\omega e^{i\varepsilon}) \quad \text{and} \quad \lim_{\varepsilon \to 0-} \sigma_L(\omega e^{i\varepsilon}), \tag{18}$$

the fact that both of these values are well-defined broad oriented skein invariants completes the proof that

COROLLARY 3. *The signature function* $\sigma_L : S^1 \to Z$ *is a broad oriented skein invariant for all links with non-zero Alexander polynomials.* □

This is an intriguing result, especially in view of the fact that $\sigma_L(\omega)$ is known to be a concordance invariant. It is natural to ask what relations there may be between skein theory and concordance theory. Another obvious question is that of what happens when the Alexander polynomial Δ_L is identically zero. In these circumstances the first Alexander ideal of the link collapses and the signature function can be thought of as extracting information about higher Alexander ideals. Kanenobu ([8] and [9]) has shown that there exist infinitely many links with identical P-polynomials but distinct second Alexander ideals, so there is no obvious reason to suppose that this information should be skein invariant. However, I know of no counter-examples to the conjecture that $\sigma_L(\omega)$ is a broad oriented skein invariant for all links.

2. GOERITZ MATRICES AND THE F-POLYNOMIAL

In this section I explore the relationships between the graph of a link, its Goeritz matrix and Kauffman's polynomial invariant $F_L(a, z)$. In particular I show that the $F(a, z)$, is essentially calculable from the Goeritz matrix of a knot. This result makes use of facts about planar graphs discovered by Whitney over 50 years ago.

2.1. THE GOERITZ MATRIX AND GRAPH OF A LINK

Kauffman [10] has defined a polynomial invariant $F_L(a, z)$ of oriented links as follows:

Recall the definition of the three *Reidemeister moves,* see Figure 3.

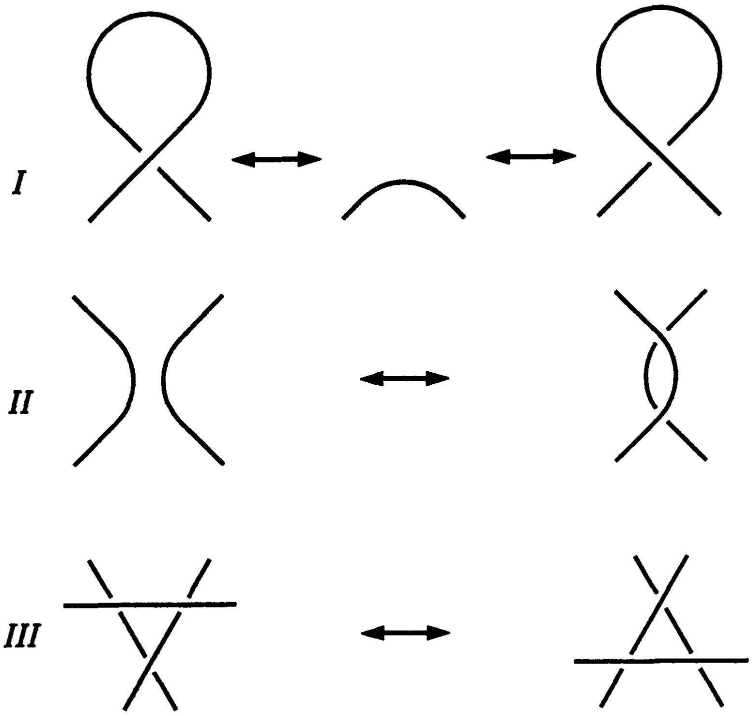


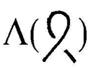
FIGURE 3

Two link diagrams represent the same link if and only if one can be transformed into the other by a finite sequence of these moves (see [18]). We define a polynomial invariant $\Lambda \in Z[a^{\pm 1}, z^{\pm 1}]$ of unoriented link diagrams by the four axioms:

i) Λ (unknot) $= 1$.

ii) Λ is invariant under Reidemeister moves II and III.

iii) The effect of Reidemeister move I on Λ is to multiply by a or a^{-1}:

(19) $\Lambda($$) = a^{-1}\Lambda($$)$, $\Lambda($$) = a\Lambda($$)$.

iv) If four link diagrams L_+, L_-, L_0 and L_∞ are identical except in a ball B where they are as shown in figure 4 then

$$\Lambda(L_+) + \Lambda(L_-) = z(\Lambda(L_0) + \Lambda(L_\infty)) \tag{20}$$

Axioms i)-iv) are sufficient to define Λ for all link diagrams. Now given an oriented diagram we can temporarily forget its orientation and calculate its Λ-polynomial. Let w be the *writhe* of the diagram (that is, the number of positive crossings less the number of negative crossings). Then

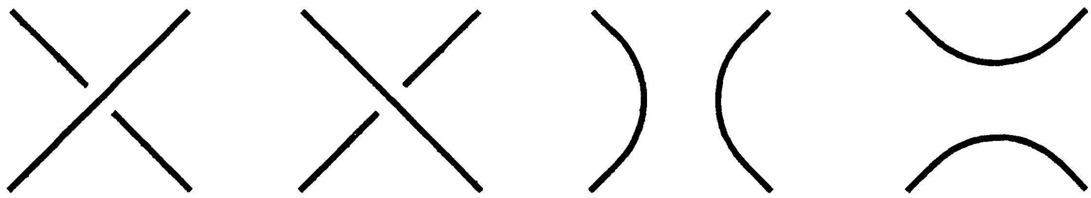
$$F(a, z) = \Lambda(a, z) \cdot a^{-w} \tag{21}$$

is a link invariant, the *Kauffman polynomial* (see [10]). Note that in order to define F_L we need the writhe, which is orientation-dependant, so F_L is an invariant of oriented links. However, for knots (1-component links), reversing the orientation leaves the sign of any given crossing, and hence the writhe, unchanged, so for knots F_L may be regarded as an unoriented invariant. If $L = \bigcup_{i=1}^{n} L_i$ is an arbitrary oriented link with components L_i, then $F_L(a, z) \cdot a^{\lambda/2}$, where $\lambda = \sum_{i \neq j} lk(L_i, L_j)$ is the *total linking number of* L is unchanged by reversal of orientations of components, and so should be regarded as an unoriented link invariant (This observation has also been made by Turaev in [23]). Like $P_L(l, m)$, $F_L(a, z)$ behaves nicely with respect to disjoint and connected sums of links:

(22) $$F_{L_1\ L_2}(a, z) = F_{L_1}(a, z) \cdot F_{L_2}(a, z)$$

(23) $$F_{L_1 \cup L_2}(a, z) = z^{-1}(a + a^{-1} - 1) \cdot F_{L_1}(a, z) \cdot F_{L_2}(a, z)$$

and is also invariant under mutation (see [1], [10]).



L_+ L_- L_0 L_∞

FIGURE 4

Now recall the definition of the *Goeritz matrix* of an unoriented connected link diagram $\mathscr{D}$ in the plane (see [6]). Such a diagram divides the plane into regions, which we proceed to colour black and white, chess board fashion, so that adjacent regions are distinct colours (It is not hard to see, using the Jordan curve theorem, that this can always be done). By convention we colour the infinite region white. Now label the black regions $R_1, R_2, \ldots, R_n$ say. At each crossing in the diagram a region R_i meets a region R_j, not necessarily distinct. This crossing takes one of two forms, illustrated in Figure 5, and we allocate signs $\xi = \pm 1$ to the crossings accordingly (The value ξ, which is defined only in the presence of a chess-board colouring of a link diagram, should not be confused with the sign of a crossing as defined in section 1. Unfortunately, the word "sign" is now well-established for each of these values in the literature.)

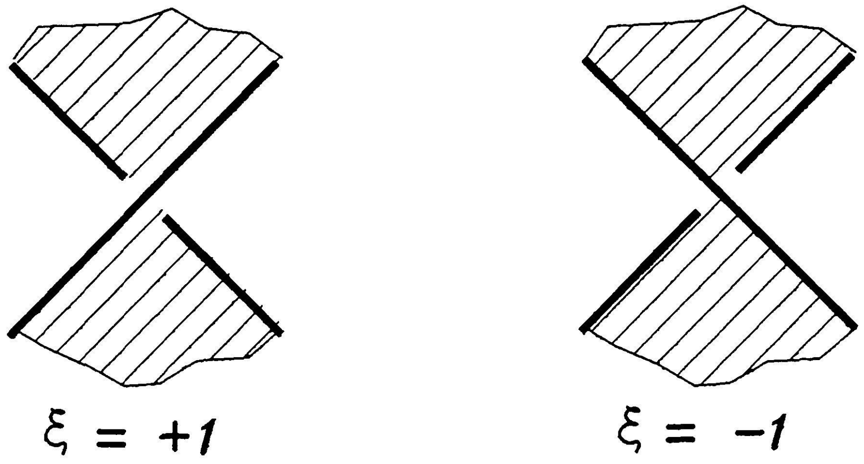


FIGURE 5

Now construct an $(n \times n)$ matrix A as follows: for $i \neq j$, let $a_{ij} = \sum \xi(c)$, where the sum is over all crossings c at which R_i meets R_j in the diagram. The diagonal elements are given by $a_{ii} = -\sum_{i \neq j} a_{ij}$ so that the row and column sums are all zero. A Goeritz matrix G for the link is then obtained by discarding the first row and column of A. Clearly G is not an invariant of the link, or even of the link diagram (any other row or column could have been discarded instead of the first one, for example). It is, however, a relation matrix for $H_1(D_L)$, where D_L is the two-fold branched cover of the link complement, and certain functions of it are true link invariants. For instance, the absolute value of its determinant is the absolute value of the determinant of the link. Further, G^{-1} is a matrix of the linking form on $H_1(D_L)$. I shall show later in this section that, up to the writhe and total linking number, Kauffman's two-variable polynomial $F_L(a, z)$ is a function of G. This raises the (unanswered) question of precisely what $F_L(a, z)$ has to do with, for example, this linking form.

The *graph* of a unoriented link diagram is constructed in a similar way. Take a vertex v_i in each black region R_i of the chess board coloured link diagram. Now for each crossing c at which R_i meets R_j, add an edge joining the corresponding vertices v_i, v_j. This edge is labelled with the sign $\xi(c)$ of the crossing. This construction provides us with a (signed) planar graph with a particular planar embedding. Conversely, given a planar embedding of a signed graph, one can construct a corresponding link diagram by placing a crossing of the appropriate sign in the middle of each edge and connecting these by arcs that run parallel to the edges of the graphs until they meet in neighbourhoods of the vertices. The graph is connected if and only if the link diagram is. See Figure 6 for an example and [2] for more details.

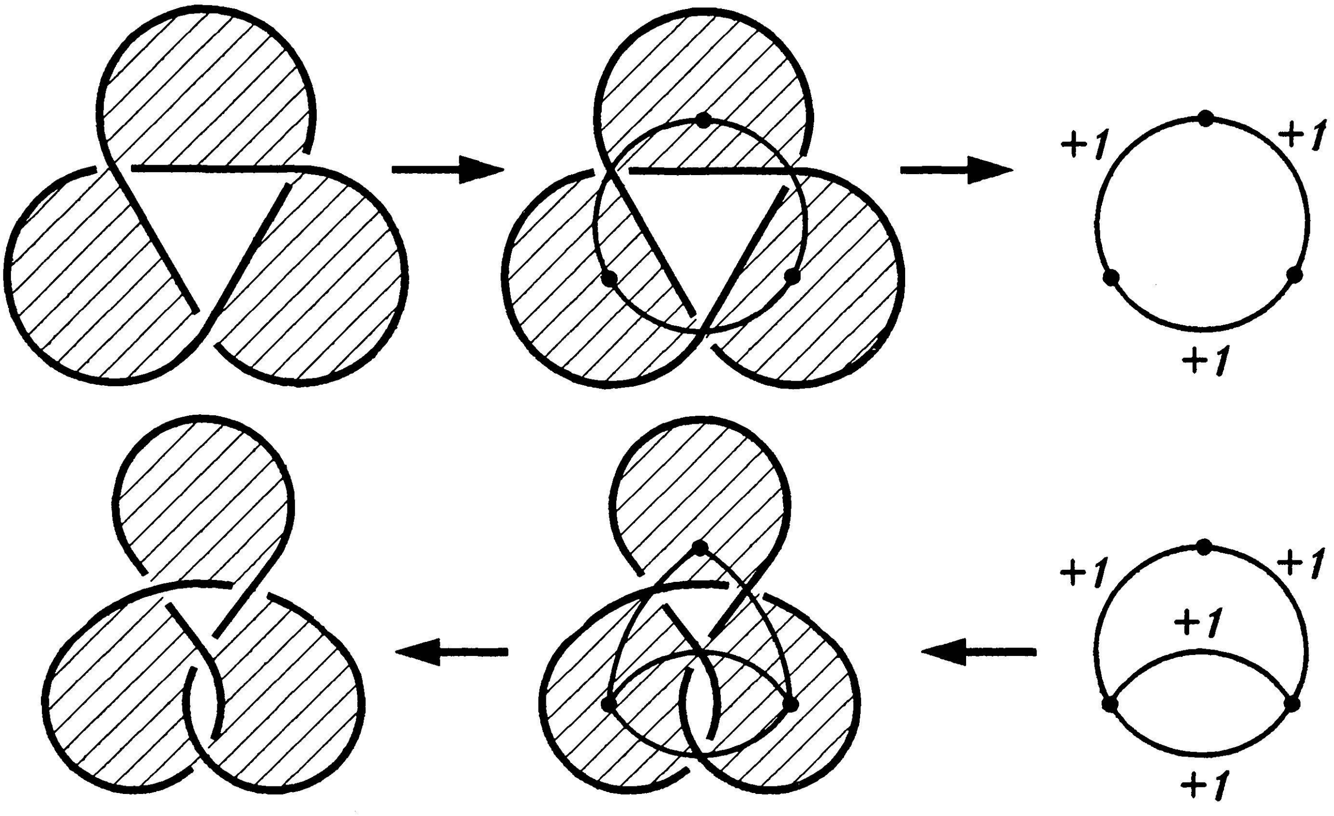


FIGURE 6

Notice that the graph of a connected diagram contains strictly more information than the Goeritz matrix, all information about the particular planar embedding and about loops in the graph being lost. Indeed, we can make use of this to construct diagrams of distinct links with identical Goeritz matrices, by picking graphs with more than one planar embedding. However, the variation that occurs here can be kept under tight control and I will make use of this fact later in this section.

2.2. KAUFFMAN'S POLYNOMIAL AND THE GOERITZ MATRIX

I now proceed to the main result of this section, linking the Goeritz matrix with Kauffman's F-polynomial invariant. Recall the observation made in section I that $\tilde{F}_L(a, z) = F_L(a, z) \cdot a^{\lambda/2}$ is invariant under change of orientation of components of L (where λ is defined to be the total linking number of L). Equivalently, we can define $\tilde{F}_L(a, z)$ by

$$\tilde{F}_L(a, z) = \Lambda_{\mathscr{D}}(a, z) \cdot a^{-w'} \tag{24}$$

where $\mathscr{D}$ is a diagram of the link L and w' is the *proper writhe* of $\mathscr{D}$, defined to be the algebraic sum of the signs of all crossings where a component of L meets itself. Note that the sign of such a crossing can be