1. Introduction

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TOPOLOGICAL SERIES OF ISOLATED PLANE CURVE SINGULARITIES

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ABSTRACT. For plane curve singularities, a topological definition of series of isolated singularities, based on the Milnor fibration, is given. Several topological invariants, including the spectrum, are computed.

1. INTRODUCTION

Let $f: (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ be a plane curve singularity, in other words, let f be an element of the ring of convergent power series $\mathbf{C}\{x, y\}$. Assume $f \neq 0$. Because $\mathbf{C}\{x, y\}$ is factorial, we can write $f = f_1^{m_1} \cdots f_r^{m_r}$ with all f_i irreducible and whenever $i \neq j$, there is no unit u with $f_i = uf_j$. The *branches* of f are the curves $f_i(x, y) = 0$.

It is well-known that for $\varepsilon > 0$ small, the intersection $L = f^{-1}(0) \cap S_\varepsilon^3$ of the curve X: $f = 0$ and a small 3-sphere of radius ε is a *link*, consisting of r components corresponding to the branches of f, and that this link determines the topological type of f (or of X). Moreover, the map $f/|f|: S_\varepsilon^3 \backslash L \to S^1$ is a fibration, called the *Milnor fibration*.

It is natural to consider L as a *multilink*, i.e. a link with integral multiplicities assigned to each component. We use the notation $L = m_1 S_1 + \cdots + m_r S_r$, where $S_i = f_i^{-1}(0) \cap S_\varepsilon^3$. These multiplicities reflect in the behaviour of the Milnor fibre F (i.e. a typical fibre of the Milnor fibration, which is a Seifert surface bounded by L) near S_i: F approaches S_i from m_i directions (see [EN]).

The Milnor fibration is important in our discussion of *topological series of isolated singularities.* A striking feature of Arnol'd's series $A, D, E, J,$ etc. (see [AGV]), is that they are somehow related to a non-isolated singularity. For example: $D_k: xy^2 + x^{k-1}$ is related to $D_\infty: xy^2$ and $Y_{r,s}: x^2y^2 + x^{r+4} + y^{s+4}$ to $Y_{\infty,\infty}: x^2y^2$. This relationship is still not completely understood.

In this paper we give (for plane curve singularities) a topological definition of series (definition 3.1), as follows. A singularity belongs to the topological series of a certain non-isolated singularity f, if its Milnor fibration arises from that of f by removing tubular neighbourhoods of the multiple components and putting something back in such a way that the result is the Milnor fibration of an isolated singularity.

With this definition in hand, we first investigate which isolated singularities belong to the series associated to a given non-isolated singularity. For example, it follows from theorem 3.4 that $D_k (k \geqslant 4)$, is the only possibility when we start with D_∞ (cf. [AGV], p. 243).

What interests us most is how the topology behaves within the series and with regard to the non-isolated singularity. We compute the Milnor number, the characteristic polynomial of the monodromy, and the spectrum of the series. For example, we will find in proposition 5.2, that the Milnor number of a series belonging to a singularity with transversal type A_1 increases linearly with steps of one, just as in the familiar case of the Arnol'd series. Many of these topological invariants have already been considered in the case of series of the form $f + \varepsilon l^k$, with l a general linear function. This was initiated by Iomdin (see [Lê]). But observe that in general such a series is a very small subseries of our topological series belonging to f.

In the last section we consider the question what we have to add to f to get a required element of the series. For instance, to $W^{\#}_{1,\infty}$: $(y^2-x^3)^2$, one may add $x^{4+q}y$ and $x^{3+q}y^2$ for $q \geqslant 1$ to obtain the whole series $W^{\#}_{1,p}$. In the case that f has only transversal A_1 singularities, we obtain explicit conditions (theorem 6.5), mainly involving intersection properties.

We use the link L of f to describe the topological type. There is a nice notation for *algebraic links* (i.e. links arising as the link of a plane curve singularity) by means of graphs that we will call *EN-diagrams* after D. Eisenbud and W.D. Neumann, who developed these graphs in [EN].

The EN-diagrams and the underlying concept of *splicing*, which is due to Siebenmann and studied extensively in [EN], are used to state our results and proofs. For example, the definition of topological series is very clear in these terms: the corresponding non-isolated singularity is visible as a subdiagram of the diagram of the series. We will only recall the main points of splicing and EN-diagrams in the next section. For details we refer to [EN] and [Ne], where one can also find how to compute several familiar topological invariants from the EN-diagram. A method of computing the *spectrum* and a splice formula for spectra are of independent interest and they are given in section 4. We will show that the spectrum of a singularity is "almost additive" under splicing.

Our definition of topological series presents a natural idea behind couterexamples (found by J. Steenbrink and J. Stevens) to the spectrum conjecture (the spectrum determines the topology of a plane curve singularity) and the — equivalent — conjecture involving the real Seifert form (cf. [SSS]). Also, A. Neméthi used the idea of topological series to define his topological trivial series [Nm].

In the Appendix, we have included the EN-diagrams and some invariants of the Arnol'd series.

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2. Splicing and series

2.1. It is clear that singularities occur in series. The simplest series have been given names, such as A, D, J, etc., by Arnol'd. But how to define a series is unclear. One looked at deformation properties such as adjacencies, etc., because the goal is to define what a series means analytically. A proper analytical description can be given for series of the form $f + l^k$, where l is a sufficiently general linear form, see the work of Iomdin and Lê, [Lê]. But already in the case of Arnol'd's series, one finds that they are not of the 'Iomdin-type'. Some series are multi-indexed, such as

$$Y_{r,s}\colon x^2y^2 + x^{r+4} + y^{s+4} ,$$

and others, such as $W^{\#}$:

$$\begin{aligned} W^{\#}_{1,2q-1}&\colon\ (y^2-x^3)^2 + x^{4+q}y \\ W^{\#}_{1,2q}&\colon\ (y^2-x^3)^2 + x^{3+q}y^2 \end{aligned}$$

make smaller steps than a linear series.

However, the most apparent properties that hold a series together, are the topological invariants. For example, the Milnor number within Arnol'd's series, increases with steps of 1. Therefore it is worthwhile to go not as far as an analytical definition, but to look for a topological one.

Another property is that, as already mentioned in the Introduction, series of isolated singularities are clearly related to non-isolated singularities, and that the hierarchy of these non-isolated singularities reflects the hierarchy of the isolated singularities. This relationship is also not completely understood. Our topological definition, which works for plane curve singularities, makes clear which isolated singularities belong to the series of a given non-isolated singularity.