Appendix Trajectories of unipotent flows and minimal sets

Objekttyp: **Appendix**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **36 (1990)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

As yet it does not seem that these results would be accessible by elementary arguments.

The study of flows on homogeneous spaces leads also to various other number theoretic results, which we shall not go into here. We refer the reader to the survey articles [4] and [19] for some of the ideas involved.

APPENDIX

TRAJECTORIES OF UNIPOTENT FLOWS AND MINIMAL SETS

We prove here a 'qualitative version' of Theorem 1.1 of [7] and use it to deduce the general case of Proposition 7. We also deduce a result used in the proof of Proposition 9. The proof of the 'qualitative version', namely Theorem A.1 below is in the same spirit at that of Theorem 2.1 of [7] and the earlier related results in [16], [3] and [5]. But the exposition here is simpler, especially on account of the weaker formulation.

We begin by setting up some notation. As before we denote by $\mathbf{R}^n$, $n \geqslant 2$, the n-dimensional vector space of n-rowed column vectors with entries in $\mathbf{R}$, by $e_1, \cdots, e_n$ the standard basis of $\mathbf{R}^n$ and by $\mathbf{Z}^n$ the subgroup generated by $\{e_1, \cdots, e_n\}$. By a lattice in $\mathbf{R}^n$ we mean a subgroup generated by n linearly independent elements in $\mathbf{R}^n$; a discrete subgroup Δ of $\mathbf{R}^n$ is a lattice if and only if $\mathbf{R}^n/\Delta$ is compact. (Cf. [13], Ch. I, §3, Theorem 2.)

We equip $\mathbf{R}^n$ with the usual inner product $< \, , \, >$ with $e_1, \cdots, e_n$ as an orthonormal basis, and the corresponding norm $\| \cdot \|$. This induces an inner product on each (vector) subspace of $\mathbf{R}^n$. For any subgroup Δ of $\mathbf{R}^n$ we denote by $\Delta_\mathbf{R}$ the subspace of $\mathbf{R}$ spanned by Δ. Let Δ be a discrete subgroup of $\mathbf{R}^n$. Then there exists a basis $x_1, \cdots, x_r$, where $r =$ dimension of $\Delta_\mathbf{R}$, such that Δ is generated by $\{x_1, \cdots, x_r\}$ (cf. [13], Ch. I, §3, Theorem 2). Let τ be a linear transformation of $\Delta_\mathbf{R}$ such that $\tau^{-1}x_1, \cdots, \tau^{-1}x_r$ is an orthonormal basis of $\Delta_\mathbf{R}$, with respect to the induced inner product. The number $|\det \tau|$ is independent of the choice of the basis $x_1, \cdots, x_r$ and the linear transformation τ, so long as the above conditions are satisfied; the number is called the determinant of Δ and is denoted by $d(\Delta)$.

As usual let $SL(n, \mathbf{R})$ be the group of $n \times n$ matrices with entries in $\mathbf{R}$ and determinant 1. By a unipotent one-parameter subgroup of $SL(n, \mathbf{R})$ we mean a unipotent one-parameter group of $n \times n$ matrices (-they are clearly contained in $SL(n, \mathbf{R})$.) We now state the theorem on orbits of lattices under unipotent one-parameter subgroups, needed in the proofs of Propositions 7 and 9.

A.1. THEOREM. *Let* $n \geqslant 2$ *be fixed. Then for* $\sigma > 0$ *there exists a* $\delta > 0$ *such that for any lattice* Λ *in* $\mathbf{R}^n$, *any unipotent one-parameter subgroup* $\{u_t\}_{t\in\mathbf{R}}$ *of* $SL(n, \mathbf{R})$ *and any* $T \geqslant 0$ *either there exists* $s \geqslant T$ *such that* $\| u_s x \| \geqslant \delta$ *for all* $x \in \Lambda - \{0\}$ *or there exists a nonzero (discrete) subgroup* Δ *of* Λ *such that* $d(u_t\Delta) < \sigma$ *for all* $t \in [0, T]$.

We introduce some more notation and prove some preliminary results before going to the proof of the theorem. For any lattice Λ in $\mathbf{R}^n$ we denote by $\mathcal{S}(\Lambda)$ the set of all nonzero subgroups of the form $\Lambda \cap W$, where W is a (vector) subspace of $\mathbf{R}^n$; such a subgroup is called a complete subgroup of Λ. For each lattice Λ we equip $\mathcal{S}(\Lambda)$ with the partial order given by the inclusion relation on subgroups and for any totally ordered subset S of $\mathcal{S}(\Lambda)$ define

$$\mathcal{L}(S, \Lambda) = \{\Delta \in \mathcal{S}(\Lambda) - S \,|\, S \cup \{\Delta\} \text{ is a totally ordered subset}\}\,;$$

the subgroups belonging to $\mathcal{L}(S, \Lambda)$ are said to be compatible with S.

We next observe some properties of the function d on class of discrete subgroups of $\mathbf{R}^n$. It is easy to see that if Δ is a discrete subgroup generated by r linearly independent elements $x_1, \cdots, x_r$ then the determinant of the $r \times r$ matrix $(\langle x_i, x_j \rangle)$ (with $\langle x_i, x_j \rangle$ in the i th row and j th column) is $d^2(\Delta)$. Under the same conditions, $d^2(\Delta)$ also coincides with the sum of squares of the determinants of all $r \times r$ minors of the $n \times r$ matrix with $x_1, \cdots, x_r$ as its columns. This may be verified either directly or using exterior products (if the reader would wish to save trouble, it may be mentioned here that Propositions 7 and 9 involve the contents of the Appendix and in particular these observations only for $n = 3$). These characterisations enable us to deduce various properties of d needed in the sequel.

A.2. LEMMA. a) *For any lattice* Λ *in* $\mathbf{R}^n$ *and any* $\rho > 0$ *the set* $\{\Delta \in \mathcal{S}(\Lambda) | d(\Lambda) < \rho\}$ *is finite.*

b) *Let* Δ *be a discrete subgroup of* $\mathbf{R}^n$. *Let* $x \in \mathbf{R}^n - \Delta_{\mathbf{R}}$ *and let* Δ' *be the (discrete) subgroup generated by* Δ *and* x. *Then* $d(\Delta') \leqslant \| x \| \, d(\Delta)$.

Proof. a) Clearly, for any nonsingular matrix g there exist constants a and b such that for any discrete subgroup Δ, $ad(\Delta) \leqslant d(g\Delta) \leqslant bd(\Delta)$. Since any lattice is of the form $g\mathbf{Z}^n$ for some nonsingular matrix g, this shows that it is enough to prove a) for $\Lambda = \mathbf{Z}^n$. If Δ is a subgroup of $\mathbf{Z}^n$ generated by r linearly independent elements $x_1, \cdots, x_r$, then the determinants of all $r \times r$ minors of the $n \times r$ matrix with columns $x_1, \cdots, x_r$ are integers. The condition $d(\Delta) < \rho$ then implies, by one of the characterisations of d, that there are only finitely many possibilities for the values of the determinants of the minors. The finiteness assertion in the Lemma therefore follows from the fact that if the corresponding $r \times r$ minors of two $n \times r$ matrices ξ and η have same determinants then the columns of ξ and η span the same subspace of $\mathbf{R}^n$.

ii) This is obvious, for instance, from the characterisation of $d(\Delta)$ in terms of the determinants of $r \times r$ minors of the $n \times r$ matrix whose columns are linearly independent and generate Δ.

A.3. LEMMA. *Let Δ be a nonzero discrete subgroup of $\mathbf{R}^n$ and let $\{u_t\}$ be a unipotent one-parameter subgroup of $SL(n, \mathbf{R})$. Then $d^2(u_t\Delta)$ is a polynomial in t of degree at most $2n(n-1)$. Further, $d(u_t\Delta)$ is constant (that is, $d(u_t\Delta) = d(\Delta)$ for all $t \in \mathbf{R}$) if and only if $\Delta_{\mathbf{R}}$ is $\{u_t\}$-invariant (that is $u_t\Delta_{\mathbf{R}} = \Delta_{\mathbf{R}}$ for all $t \in \mathbf{R}$).*

Proof. If ν is a $n \times n$ nilpotent matrix then by the Jordan canonical form $\nu^n = 0$. This implies that for any unipotent one-parameter subgroup $\{u_t\}$ of $SL(n, \mathbf{R})$ and any $x \in \mathbf{R}^n$, the coordinates (entries) of $u_t x$ are polynomials in t of degree at most $n - 1$. Now let Δ be a discrete subgroup generated by r linearly independent elements $x_1, \cdots, x_r$. Then $d^2(u_t\Delta)$ is the determinant of the $r \times r$ matrix $(< u_t x_i, u_t x_j >)$. By the preceding remark each entry $< u_t x_i, u_t x_j >$ is a polynomial in t of degree at most $2(n-1)$. Hence the determinant is a polynomial of degree at most $2n(n-1)$.

Next let Δ be a discrete subgroup such that $d(u_t\Delta) = d(\Delta)$ for all $t \in \mathbf{R}$. Let $x_1, \cdots, x_r$ be linearly independent elements generating Δ. The determinant of each $r \times r$ minor of the $n \times r$ matrix with columns $u_t x_1, \cdots, u_t x_r$ is a polynomial in t. Since sum of squares of these is $d^2(u_t\Delta) = d^2(\Delta)$ for all $t \in \mathbf{R}$, it follows that each of them is constant. Thus for each $t \in \mathbf{R}$ any $r \times r$ minor of the $n \times r$ matrix with columns $u_t x_1, \cdots, u_t x_r$ has the same determinant as the corresponding minor in the $n \times r$ matrix with columns $x_1, \cdots, x_r$. This implies that for any t, $u_t x_1, \cdots, u_t x_r$ span the same subspace as $x_1, \cdots, x_r$, or equivalently $u_t\Delta_{\mathbf{R}} = \Delta_{\mathbf{R}}$. This proves the Lemma.

For any $m \in \mathbf{N}$ we denote by $\mathscr{P}_m$ the set of all nonnegative polynomials of degree at most m; 'nonnegative' refers to the values being nonnegative — some of the coefficients could be negative. For the proof of Theorem 8 we need the following simple properties of nonnegative polynomials.

A.4. LEMMA. a) *For any $m \in \mathbf{N}$ and $\lambda > 1$ there exists $\varepsilon > 0$ such that the following holds: if $P \in \mathscr{P}_m$ and there exists $s \in [0, 1]$ such that $P(s) \geqslant 1$ and $P(1) < \varepsilon$ then there exists $t \in [1, \lambda]$ such that $P(t) = \varepsilon$.*

b) *For any* $m \in \mathbf{N}$ *and* $\mu > 1$ *there exist constants* $\varepsilon_1, \varepsilon_2 > 0$ *such that the following holds: if* $P \in \mathscr{P}_m, P(s) \leqslant 1$ *for all* $s \in [0, 1]$ *and* $P(1) = 1$ *then there exists* $i, 0 \leqslant i \leqslant m,$ *such that* $\varepsilon_1 \leqslant P(t) \leqslant \varepsilon_2$ *for all* $t \in [\mu^{2i+1}, \mu^{2i+2}]$.

Proof. It can be seen that given an interval I of positive length and a $c > 0$ there exists a constant M such that any $P \in \mathscr{P}_m$ such that $P(t) \leqslant c$ for all $t \in I$, has all the coefficients of absolute value at most M; in particular, any sequence of polynomials bounded by c on I has a subsequence converging to a polynomial in $\mathscr{P}_m$. Now if a) does not hold there must exist a sequence $\{P_k\}$ in $\mathscr{P}_m$ such that $P_k(t) \to 0$ uniformly on $[1, \lambda]$ but the supremum of each P_k on $[0, 1]$ is at least 1; this is impossible by the above observation. To prove b) we first observe that existence of the upper bound ε_2 follows from the bound on the coefficients as above, when we take $I = [0, 1]$ and $c = 1$. Thus if b) does not hold there exists a sequence $\{P_k\}$ in $\mathscr{P}_m$ such that for each $k, P_k(s) \leqslant 1$ for all $s \in [0, 1], P_k(1) = 1$ and $\inf\{P_k(t) \mid t \in [\mu^{2i+1}, \mu^{2i+2}]\} \to 0$ as $k \to \infty$, for each $i = 0, \cdots, m$; this is impossible since the limit of any subsequence would be a nontrivial polynomial in $\mathscr{P}_m$ with at least $m + 1$ zeros.

For the rest of the argument we fix some constants as follows: Let $n \in \mathbf{N}$ and $\mu > 1$ be arbitrary. Let $m = 2n^2$ and $\lambda > 1$ be such that $(\lambda - 1) \leqslant (\mu - 1)/\mu^{2m+2}$. Let $0 < \alpha < 1$ be such that condition a) as in Lemma A.4 holds for $\varepsilon = \alpha^2$ with m and λ as above and let $0 < \beta_1 < 1 < \beta_2$ be such that condition b) of Lemma A.4 holds for $\varepsilon_1 = \beta_1^2$ and $\varepsilon_2 = \beta_2^2$ with m and μ as above.

A.5. PROPOSITION. *Let* $\{u_t\}$ *be a unipotent one-parameter subgroup of* $SL(n, \mathbf{R}), \Lambda$ *be a lattice in* $\mathbf{R}^n$ *and* S *be a totally ordered subset of* $\mathscr{S}(\Lambda)$. *Let* $\tau > 0$ *and* $T \geqslant 0$ *be such that for each* $\Phi \in \mathscr{L}(S, \Lambda)$ *there exists a* $t \in [0, T]$ *such that* $d(u_t\Phi) \geqslant \tau$. *Then either* $d(u_T\Phi) \geqslant \alpha\tau$ *for all* $\Phi \in \mathscr{L}(S, \Lambda)$ *or there exist a* $\Delta \in \mathscr{L}(S, \Lambda)$ *and a* $T_1 \in [T, (2 - \mu^{-1})T]$ *such that the following conditions are satisfied:*

i) $\tau\alpha\beta_1 \leqslant d(u_t\Delta) \leqslant \tau\alpha\beta_2$ *for all* $t \in [T_1, T + \mu(T_1 - T)]$

ii) *for each* $\Phi \in \mathscr{L}(S, \Lambda)$ *there exists* $t \in [T, T_1]$ *such that* $d(u_t\Phi) \geqslant \alpha\tau$.

Proof. Let $\mathscr{F} = \{\Phi \in \mathscr{L}(S, \Lambda) \mid d(u_T\Phi) < \alpha\tau\}$. If $\mathscr{F}$ is empty then we are through. Now suppose that $\mathscr{F}$ is nonempty. By Lemma A.2 a) $\mathscr{F}$ is finite; say $\mathscr{F} = \{\Phi_1, \cdots, \Phi_q\}$, where $q \geqslant 1$. For each $j, 1 \leqslant j \leqslant q,$ we choose

$t_j \in [T, \lambda T]$ as follows: Observe that $d(u_T\Phi_j) < \alpha\tau$ and that there exists, by hypothesis, a $t \in [0, T]$ such that $d(u_t\Phi_j) \geqslant \tau$. Hence applying Lemma A.4, a) to the polynomial $t \mapsto d^2(u_{tT}\Phi_j)/\tau^2$ we conclude that there exists a $t_j \in [T, \lambda T]$ such that $d(u_{t_j}\Phi_j) = \alpha\tau$; taking the smallest such number we may also assume t_j to have the further property that $d(u_t\Phi_j) \leqslant \alpha\tau$ for all $t \in [T, t_j]$.

Next let $1 \leqslant k \leqslant q$ be such that $t_j \leqslant t_k$ for all $1 \leqslant j \leqslant q$. We choose $\Delta = \Phi_k$. Then we have $d(u_t\Delta) \leqslant \alpha\tau$ for all $t \in [T, t_k]$ and $d(u_{t_k}\Delta) = \alpha\tau$. Hence by Lemma A.4 b), applied to the polynomial $t \mapsto d^2(u_{(t_k - T)t + T}\,\Delta)/\alpha^2\tau^2$, it follows that there exists an i such that $0 \leqslant i \leqslant m$ and

$$(*) \qquad \tau\alpha\beta_1 \leqslant d(u_t\Delta) \leqslant \tau\alpha\beta_2 \quad \text{for all} \quad t \in [T_1, T_2]\ ,$$

where $T_1 = T + \mu^{2i+1}(t_k - T)$ and $T_2 = T + \mu^{2i+2}(t_k - T)$. Then

$$\begin{aligned} T + \mu(T_1 - T) \ &= \ T + \mu^{2i+2}(t_k - T) \ \leqslant \ T + \mu^{2m+2}(t_k - T) \\ &\leqslant \ T + \mu^{2m+2}(\lambda - 1)T \ \leqslant \ \mu T\,, \end{aligned}$$

since $i \leqslant m$, $t_k \in [T, \lambda T]$ and $(\lambda - 1) \leqslant (\mu - 1)/\mu^{2m+2}$. This shows that $T_1 \in [T, (2 - \mu^{-1})T]$. Also $(*)$ shows that condition i) as in the Proposition is satisfied for Δ. Condition ii) is obvious from the construction; if $\Phi \notin \mathscr{F}$ then $d(u_T\Phi) \geqslant \alpha\tau$ and if $\Phi \in \mathscr{F}$, say $\Phi = \Phi_j$ where $1 \leqslant j \leqslant q$, then we have $T \leqslant t_j \leqslant t_k \leqslant T_1$ and $d(u_{t_j}\Phi_j) = \alpha\tau$, which verifies the condition for all $\Phi \in \mathscr{L}(S, \Lambda)$. Hence the Proposition.

A.6. COROLLARY. *Let $\{u_t\}$, Λ, S, $\tau > 0$ and $T \geqslant 0$ be as in Proposition A.5. Let p be the cardinality of S. Then there exist a totally ordered subset M of $\mathscr{S}(\Lambda)$ containing S and a $R \in [T, \mu T]$ such that the following conditions are satisfied:*

1) $\alpha^{(n-p)}\beta_1\tau \leqslant d(u_R\Phi) \leqslant \alpha\beta_2\tau$ *for all* $\Phi \in M - S$
2) $d(u_R\Phi) \geqslant \alpha^{(n-p)}\tau$ *for all* $\Phi \in \mathscr{L}(M, \Lambda)$.

Proof. We proceed by induction on $(n - p)$. If $p = n$ then S is a maximal totally ordered subset (so $\mathscr{L}(S, \Lambda)$ is empty) and the desired assertion holds for $M = S$. We now assume the result for $p + 1$ in the place of p and consider Λ, S, τ and T as in the hypothesis. If $d(u_T\Phi) \geqslant \alpha\tau$ for $\Phi \in \mathscr{L}(S, \Lambda)$ then we can choose $M = S$ and $R = T$. If not, then by Proposition A.5 there exist $\Delta \in \mathscr{L}(S, \Lambda)$ and $T_1 \in [T, (2 - \mu^{-1})T]$ such that $\tau\alpha\beta_1 \leqslant d(u_t\Delta) \leqslant \tau\alpha\beta_2$ for all $t \in [T_1, T + \mu(T_1 - T)]$ and for each $\Phi \in \mathscr{L}(S, \Lambda)$ there exists a $t \in [T, T_1]$

such that $d(u_t\Phi) \geqslant \alpha\tau$. Put $\Lambda_1 = u_T\Lambda$, $S_1 = \{u_T\Phi \mid \Phi = \Delta$ or $\Phi \in S\}$ and $\tau_1 = \alpha\tau$. Then Λ_1 is a lattice in $\mathbf{R}^n$, S_1 is a totally ordered subset of $\mathscr{S}(\Lambda_1)$ and the second part of the preceding conclusion implies that the hypothesis of the corollary applies to Λ_1, S_1, τ_1 and $T_1 - T$ in the place of Λ, S, τ and T respectively; we note that any $\Psi \in \mathscr{L}(S_1, \Lambda_1)$ is of the form $u_T\Phi$, $\Phi \in \mathscr{L}(S, \Lambda)$. Hence by the induction hypothesis there exist a subset M_1 of $\mathscr{S}(\Lambda_1)$ containing S_1 and a $R_1 \in [T_1 - T, \mu(T_1 - T)]$ such that $\alpha^{(n-p-1)}\beta_1\tau_1 \leqslant d(u_{R_1}\Delta_1) \leqslant \alpha\beta_2\tau_1$ for all $\Delta_1 \in M_1 - S_1$ and $d(u_{R_1}\Phi) \geqslant \alpha^{(n-p-1)}\tau_1$ for all $\Phi \in \mathscr{L}(M_1, \Lambda_1)$. Put $M = \{u_{-T}\Delta_1 \mid \Delta_1 \in M_1\}$ and $R = T + R_1$. Then $T \leqslant R \leqslant T + \mu(T_1 - T) \leqslant \mu T$, since $T_1 \in [T, (2-\mu^{-1})T]$. Observe that $M - S = \{\Phi \mid \Phi = \Delta$ or $u_T\Phi \in M_1 - S_1\}$. The choice of Δ, using Proposition A.5 shows that Condition 1) in the conclusion of the Corollary holds for $\Phi = \Delta$. If $u_T\Phi \in M_1 - S_1$ then we have $d(u_R\Phi) = d(u_{R_1}u_T\Phi) \in [\alpha^{(n-p-1)}\beta_1\tau_1, \alpha\beta_2\tau_1] \subset [\alpha^{(n-p)}\beta_1\tau, \alpha\beta_2\tau]$, since $\tau_1 = \alpha\tau$ and $\alpha < 1$. Thus Condition 1) holds for all $\Phi \in M - S$. For $\Phi \in \mathscr{L}(M, \Lambda)$ we have $d(u_R\Phi) = d(u_{R_1}u_T\Phi) \geqslant \alpha^{(n-p-1)}\tau_1 = \alpha^{(n-p)}\tau$, since $u_T\Phi \in \mathscr{L}(M, \Lambda_1)$ and $\tau_1 = \alpha\tau$; this shows that Condition 2) is also satisfied. This proves the Corollary.

Proof of Theorem A.1. Let n and σ be as in the hypothesis of the theorem. Let $\mu > 1$ be chosen arbitrarily and let α, β_1 and β_2 be the constants chosen ahead of Proposition A.5, depending on n and μ; recall that $0 < \alpha < 1$ and $0 < \beta_1 < 1 < \beta_2$. Let $t = \min\{\sigma, \sigma^{-1}\}$ and let $\delta = \alpha^n\beta_1\beta_2^{-1}\tau$.

Now let $\{u_t\}$ be any unipotent one-parameter subgroup of $SL(n, \mathbf{R})$, Λ be any lattice in $\mathbf{R}^n$ and let $T \geqslant 0$ be such that there does not exist any nonzero subgroup Δ of Λ such that $d(u_t\Delta) < \sigma$ for all $t \in [0, T]$. This implies that for all $\Phi \in \mathscr{S}(\Lambda)$ there exists $\tau \in [0, T]$ such that $d(u_t\Phi) \geqslant \sigma \geqslant \tau$. In other words, the condition in the Corollary holds if we choose S to be the empty subset. Hence by the Corollay there exists a totally ordered subset M of $\mathscr{S}(\Lambda)$ and a $R \in [T, \mu T]$ such that $\alpha^n\beta_1\tau \leqslant d(u_R\Phi) \leqslant \alpha\beta_2\tau \leqslant \beta_2$ for all $\Phi \in M$ and $d(u_R\Phi) \geqslant \alpha^n\tau$ for all $\Phi \in \mathscr{L}(M, \Lambda)$. Now let x be any primitive element in Λ and let Δ be the subgroup generated by x. Then $\Delta \in \mathscr{S}(\Lambda)$. If x is contained in every element of M then we see that $\Delta \in M \cup \mathscr{L}(M, \Lambda)$ and hence $\| u_Rx \| = d(u_R\Delta) \geqslant \alpha^n\beta_1\tau \geqslant \delta$. Now suppose that x is not contained in some elements of M and let Φ be the largest element of M not containing x. Let Ψ be the smallest complete subgroup of Λ (element of $\mathscr{S}(\Lambda)$) containing Φ and x. Then we see that $\Psi \in M \cup \mathscr{L}(M, \Lambda)$, as every element of M containing Φ as a proper subgroup also contains x. Now, by Lemma A.2 b) $d(u_R\Psi) \leqslant \| u_Rx \| \, d(u_R\Phi)$. But since $\Phi \in M$ and $\Psi \in M \cup \mathscr{L}(M, \Lambda)$ we have

$d(u_R\Phi) \leqslant \beta_2$ and $d(u_R\Psi) \geqslant \alpha^n\beta_1\tau$. Thus we get that $\| u_R x \| \geqslant \alpha^n\beta_1\beta_2^{-1}\tau = \delta$. Hence $\| u_R x \| \geqslant \delta$ for all primitive x in Λ and hence the same holds for all $x \in \Lambda - \{0\}$, thus proving the Theorem.

A.7. COROLLARY. *Given* $\sigma > 0$ *there exists a neighbourhood* Ω *of* 0 *in* $\mathbf{R}^n$ *such that for any unipotent one-parameter subgroup* $\{u_t\}$ *in* $SL(n, \mathbf{R})$ *and any lattice* Λ *in* $\mathbf{R}^n$ *one of the following holds:*

1) $\{t \geqslant 0 \mid u_t\Lambda \cap \Omega = (0)\}$ *is an unbounded subset of* $\mathbf{R}$.
2) *there exists a nonzero subgroup* Δ *of* Λ *such that the subspace spanned by* Δ *is* $\{u_t\}$*-invariant and* $d(u_t\Delta) = d(\Delta) < \sigma$ *for all* $t \in \mathbf{R}$.

Proof. Let $\delta > 0$ be such that Theorem A.1 holds for the given σ and let $\Omega = \{x \in \mathbf{R}^n \mid \| x \| < \delta\}$. Let $\{u_t\}$ and Λ be as in the hypothesis and suppose that Condition 1) does not hold. Then by Theorem A.1 there exists a nonzero subgroup Δ of Λ such that $d(u_t\Delta) < \sigma$ for all $t \geqslant 0$. Since $d^2(u_t\Delta)$ is a polynomial in t, this implies that $d(u_t\Delta)$ is constant; i.e. $d(u_t\Delta) = d(\Delta) < \sigma$ for all $t \in \mathbf{R}$. By Lemma A.3, this implies that the subspace $\Delta_{\mathbf{R}}$ spanned by Δ is $\{u_t\}$-invariant. This proves the corollary.

We next relate Theorem A.1 and Corollary A.7 to behaviour of orbits of unipotent one-parameter groups of $SL(n, \mathbf{R})/SL(n, \mathbf{Z})$, where $SL(n, \mathbf{Z})$ is the subgroup consisting of integral matrices. This involves the Mahler criterion (sometimes also called Mahler's selection theorem) recalled below. The reader may refer [2], [13] or [24] depending on the background; one could also consult Mahlers original paper [15].

Let $\mathscr{L}_n$ be the set of all lattices in $\mathbf{R}^n$. On $\mathscr{L}_n$ one defines a topology by prescribing that for each basis $x_1, \cdots, x_n$ of $\mathbf{R}^n$ and $\varepsilon > 0$ the set $\Omega(x_1, \cdots, x_n, \varepsilon)$, of all lattices Λ such that Λ is generated by a basis $y_1, \cdots, y_n$ of $\mathbf{R}^n$ satisfying $\| x_i - y_i \| < \varepsilon$ for all i, be open. This indeed defines a first countable Hausdorff topology on $\mathscr{L}_n$. The Mahler criterion asserts that if $\{\Lambda_i\}$ is a sequence in $\mathscr{L}_n$ and there exist c and δ such that for all i, $d(\Lambda_i) \leqslant c$ and $\| x \| \geqslant \delta$ for all $x \in \Lambda_i - \{0\}$ then $\{\Lambda_i\}$ has a convergent subsequence. The criterion implies in particular that $\mathscr{L}_n$ is locally compact.

Now let $\mathscr{U}_n$ be the subset of $\mathscr{L}_n$ consisting of all lattices of determinant 1. Then $\mathscr{U}_n$ is a closed subset, as d is continuous, and in particular it is locally compact. For each $g \in SL(n, \mathbf{R})$ and $\Lambda \in \mathscr{U}_n$, $g\Lambda \in \mathscr{U}_n$ and the map $(g, \Lambda) \mapsto g\Lambda$ defines a continuous action of $SL(n, \mathbf{R})$ on $\mathscr{U}_n$. It is easy to see that the action is transitive and that $SL(n, \mathbf{Z})$ is the isotropy subgroup of the lattice $\mathbf{Z}^n$, under the action. Hence $SL(n, \mathbf{R})/SL(n, \mathbf{Z})$, equipped with the quotient topology, is homeomorphic to $\mathscr{U}_n$ via the correspondence

$gSL(n, \mathbf{Z}) \mapsto g\mathbf{Z}^n$ for $g \in SL(n, \mathbf{R})$ (cf. [9], Ch. V, §1, Theorem 8 or [10], (1.6.1)). The Mahler criterion therefore implies that for any $\delta > 0$ the set

$$\{gSL(n, \mathbf{Z}) \mid \| gp \| \geqslant \delta \quad \text{for all} \quad p \in \mathbf{Z}^n - \{0\}\}$$

is a compact subset of $SL(n, \mathbf{R})/SL(n, \mathbf{Z})$. Theorem A.1 and Corollary A.7 therefore imply the following

A.8. THEOREM. *Let $n \geqslant 2$ be fixed. Then for any $\sigma > 0$ there exists a compact subset K of $SL(n, \mathbf{R})/SL(n, \mathbf{Z})$ such that for any $x = gSL(n, \mathbf{Z}) \in SL(n, \mathbf{R})/SL(n, \mathbf{Z})$, where $g \in G$, and any unipotent one-parameter subgroup $\{u_t\}$ of $SL(n, \mathbf{R})$ the following conditions are satisfied:*

a) *for any $T \geqslant 0$ either there exists a $t \geqslant T$ such that $u_t x \in K$ or there exists a nonzero discrete subgroup Δ of $\mathbf{Z}^n$ such that $d(u_t g\Delta) < \sigma$ for all $t \in [0, T]$,*

b) *if $\{t \geqslant 0 \mid u_t x \in K\}$ is bounded then there exists a nonzero subgroup Δ of $\mathbf{Z}^n$ such that the subspace spanned by Δ is $\{g^{-1}u_t g\}$-invariant and $d(u_t g\Delta) = d(g\Delta) < \sigma$ for all $t \in \mathbf{R}$.*

We next deduce the general case of Proposition 7, which we had deferred until proving the above theorem. We follow the notation G, Γ, V_1, DV_1 etc., as in the main part. The diagonal matrix diag $(\lambda, 1, \lambda^{-1})$ where $\lambda \in \mathbf{R}^*$ will be denoted by $a(\lambda)$, rather than $d(\lambda)$, to avoid confusion with $d(\Delta)$ for discrete subgroups Δ. Also as before we denote by e_1, e_2, e_3 the standard basis of $\mathbf{R}^3$. The subspaces spanned by $\{e_1\}$ and $\{e_1, e_2\}$ are denoted by W_1 and W_2 respectively.

We first prove part b) of Proposition 7, namely the following:

A.9. PROPOSITION. *There are no closed DV_1-orbits. Any nonempty closed DV_1-invariant subset contains a minimal nonempty closed DV_1-invariant subset.*

Proof. Let K be a compact subset of G/Γ such that the contention of Theorem A.8 holds for ($n = 3$ and) $\sigma = 1$. We first show that for any $x = g\Gamma \in G/\Gamma$, where $g \in G$, there exists $\lambda_0 > 0$ such that for all $\lambda \geqslant \lambda_0, \{t \geqslant 0 \mid v_1(t)a(\lambda)x \in K\}$ is unbounded. Let $g \in G$ be given and let $x = g\Gamma$. Define

$$\lambda_0 = \max\{1, 1/d(g\mathbf{Z}^3 \cap W_1), 1/d(g\mathbf{Z}^3 \cap W_2)\} .$$

Let $\lambda \geqslant \lambda_0$ be arbitrary. Let Δ be a nonzero discrete subgroup $\mathbf{Z}^3$ such that $\Delta_{\mathbf{R}}$ is a proper subspace invariant under the action of $g^{-1}a(\lambda)^{-1}V_1 a(\lambda)g = g^{-1}V_1 g$. Then $g\Delta_{\mathbf{R}}$ is a nonzero proper V_1-invariant subspace. A simple computation shows that W_1 and W_2 are the only such subspaces. Hence $g\Delta_{\mathbf{R}} = W_1$ or W_2. Both W_1 and W_2 are $a(\lambda)$-invariant and the determinant of the restriction of $a(\lambda)$ to either subspace is λ. Hence the preceding observation implies that $d(a(\lambda)g\Delta) = \lambda d(g\Delta)$. Since $g\Delta$ is contained in either $g\mathbf{Z}^3 \cap W_1$ or $g\mathbf{Z}^3 \cap W_2$, by the choice of λ_0 we get that $d(g\Delta) \geqslant \lambda_0^{-1}$. Hence $d(a(\lambda)g\Delta) \geqslant \lambda/\lambda_0 \geqslant 1 = \sigma$. In view of this verification for all Δ as above, Theorem A.8 b) implies that $\{t \geqslant 0 \mid v_1(t)a(\lambda)x \in K\}$ is unbounded as claimed; note that as $\sigma = 1$, the subgroup Δ in Theorem A.8 b) spans a proper subspace.

We now deduce the assertions as in the proposition. If possible let $x \in G/\Gamma$ be such that DV_1x is a closed orbit in G/Γ. Let $\Phi = \{g \in G \mid gx = x\}$. Then Φ is a discrete subgroup of DV_1 and the map $\theta\colon DV_1/\Phi \to DV_1x$ defined by $\theta(g\Phi) = gx$ for all $g \in DV_1$ is a homeomorphism (cf. [9], Ch. V, §1, Theorem 8 or [10], (1.6.1)). By Lemma 6 Φ is either contained in V_1 or it is a cyclic subgroup generated by an element of the form vdv^{-1} where $d \in D$ and $v \in V_1$. Suppose the latter possibility holds. Then we see that for each $\lambda > 0$, $V_1 a(\lambda)\Phi$ is closed and $t \mapsto v_1(t)a(\lambda)\Phi$ defines a homeomorphism of $\mathbf{R}$ onto $V_1 a(\lambda)\Phi/\Phi$. Since θ is a homeomorphism, this implies that for each $\lambda > 0$, $V_1 a(\lambda)x$ is closed and $t \mapsto v_1(t)a(\lambda)x$ is a homeomorphism of $\mathbf{R}$ onto $V_1 a(\lambda)x$. But, by our observation above, there exists λ_0 such that for $\lambda \geqslant \lambda_0$, $\{t \geqslant 0 \mid v_1(t)a(\lambda)x \in K\}$ is unbounded. This is a contradiction since by the preceding observation it implies that $\{v_1(t)a(\lambda)x \mid t \geqslant 0\} \cap K$ is a closed noncompact subset of K. Now suppose Φ is contained in V_1. Let $\{\lambda_i\}$ be a sequence of positive numbers such that $\lambda_i \to \infty$. Then we see that as $\Phi \subset V_1$, for any sequence $\{t_i\}$ in $\mathbf{R}$, $\{a(\lambda_i)v_1(t_i)\Phi\}$ has no convergent subsequence in DV_1/Φ. Since θ is a homeomorphism this implies that for any sequence $\{t_i\}$ in $\mathbf{R}$, $\{a(\lambda_i)v_1(t_i)x\}$ has no convergent subsequence. But this is a contradiction since K is compact and for all large λ there exists $t \geqslant 0$ such that $v_1(t)a(\lambda)x = a(\lambda)\left(v_1(\lambda^{-1}t)\right)x \in K$. Hence there are no closed DV_1-orbits.

Now let X be any nonempty closed DV_1-invariant subset of G/Γ. We see that if $\{X_i\}_{i\in I}$ is a totally ordered family (with respect to inclusion) of nonempty closed DV_1-invariant subsets of X (indexed by a set I), then $\cap_{i\in I} X_i$ is nonempty as it contains $\cap_{i\in I}(X_i \cap K)$ and by the above observation each $X_i \cap K$ is a nonempty compact subset. Hence by Zorn's lemma the class of all nonempty closed DV_1-invariant subsets of X has a minimal element. This proves the Proposition.

To prove the other part of Proposition 7 we need the following Lemmas.

A.10. LEMMA. *Let* $q = 1$ *or* 2 *and for any* $\rho > 0$ *let*

$$A(q, \rho) = \{g\Gamma \mid g \in G,\ g\mathbf{Z}^3 \cap W_q \ \text{ spans } \ W_q \ \text{ and } \ d(g\mathbf{Z}^3 \cap W_q) = \rho\}\,.$$

Then $A(q, \rho)$ *is a closed subset of* G/Γ.

Proof. It is straightforward to verify that any subset as in the statement can be expressed as $Q_q a\Gamma/\Gamma$ for some diagonal matrix a, Q_1 and Q_2 being the subgroups defined by

$$Q_1 = \{g \in G \mid ge_1 = e_1\} \quad \text{and} \quad Q_2 = \{g \in G \mid {}^t ge_3 = e_3\}\,.$$

Now consider the natural action of G on $\mathbf{R}^3$. We see that Γe_1 is a discrete subset of $\mathbf{R}^3$. Hence so is Γse_1 for any $s \in \mathbf{R}$. Let b be a diagonal matrix. Then $be_1 = se_1$ for some $s \in \mathbf{R}$ and hence Γbe_1 is a closed subset of $\mathbf{R}^3$. The continuity of the action and the fact that Q_1 is the subgroup consisting of all elements fixing e_1 now implies that ΓbQ_1 is a closed subset of G, for any diagonal matrix b. Hence so is $Q_1 a\Gamma = (\Gamma a^{-1} Q_1)^{-1}$, for any diagonal matrix a. This proves the case of the Lemma with $q = 1$. The case of $q = 2$ follows from a similar argument with the contragradient action, defined by $(g, p) \mapsto {}^t g^{-1} p$ for all $p \in \mathbf{R}^3$, in the place of the natural action, and e_3 in the place of e_1.

A.11. LEMMA. *Let* Z *be a locally compact space and let* $\{\varphi_t\}_{t \in R}$ *be a one-parameter group of homeomorphisms of* Z *acting continuously on* Z. *Suppose that there exists a compact subset* K *of* Z *such that for each* $z \in Z$, *the sets* $\{t \geqslant 0 \mid \varphi_t z \in K\}$ *and* $\{t \leqslant 0 \mid \varphi_t z \in K\}$ *are unbounded. Then* Z *is compact.*

Proof. Let $\varphi = \varphi_1$. Replacing K by the larger compact set $\{\varphi_s z \mid -1 \leqslant s \leqslant 1, z \in K\}$ if necessary, we may assume that for each $z \in Z$, $\{k \in \mathbf{N} \mid \varphi^k z \in K\}$ and $\{k \in \mathbf{N} \mid \varphi^{-k} z \in K\}$ are unbounded subsets of $\mathbf{N}$. Let K_1 be a compact neighbourhood of K and let $\Omega = Z - K_1$. Let $B = \cap_{j=0}^{\infty} \varphi^j \overline{\Omega}$. Then $\varphi^{-j} B \subset B \subset \overline{\Omega} \subset Z - K$ for all $j \in \mathbf{N}$ and hence the condition on K implies that B is empty. Hence φB is empty. Since K_1 is compact this implies that there exists $m \in \mathbf{N}$ such that $\cap_{j=1}^{m} \varphi^i \overline{\Omega}$ is contained in Ω. Then $\cap_{j=0}^{m} \varphi^j \Omega = \cap_{j=1}^{m} \varphi^j \Omega = E$ say. Then we see that $\varphi E \subset E$ and hence

$\varphi^j E \subset E$ for all $j \in \mathbf{N}$. Since $E \subset \Omega \subset Z - K$, the condition on K implies that E is empty. Hence $Z = \cup_{j=1}^{m} \varphi^j(Z - \Omega)$, which is compact.

Part a) of Proposition 7 now follows from the following Proposition and the earlier observation for compact invariant sets.

A.12. PROPOSITION. *Any nonempty closed V_1-invariant subset of G/Γ contains a compact nonempty V_1-invariant subset.*

Proof. Let X be a nonempty closed V_1-invariant subset of G/Γ. For $q = 1, 2$ and any $\rho > 0$ let $A(q, \rho)$ denote the closed subset of G/Γ as in Lemma A.10. In proving the Proposition, by replacing X by a smaller (nonempty) subset if necessary, we may assume that for each $q = 1, 2$ and $\rho > 0$, either $X \cap A(q, \rho) = \emptyset$ or $X \subset A(q, \rho)$; note that the sets $A(q, \rho)$ are V_1-invariant and that for each q the sets $\{A(q, \rho)\}_{\rho > 0}$ are mutually disjoint. Now let $\sigma \leqslant 1$ be such that if X is contained in $A(q, \rho)$ for some $q = 1$ or 2 and $\rho > 0$ then $\sigma \leqslant \rho$. Let K be a compact subset of G/Γ such that the contention of Theorem A.8 holds for this σ. We shall show that for each $x \in X$ the sets $\{t \geqslant 0 \mid v_1(t)x \in K\}$ and $\{t \leqslant 0 \mid v_1(t)x \in K\}$ are unbounded; by Lemma A.11 this implies that X (rather the replaced set) is compact, thus proving the proposition. Suppose for some $x \in X$, say $x = g\Gamma$ where $g \in G$, one of the sets as above is bounded. Then by Theorem A.8, applied to either $\{v_1(t)\}$ or $\{v_1(-t)\}$ in the place of $\{u_t\}$ and x as above, it follows that there exists a nonzero subgroup Δ of $\mathbf{Z}^n$ such that $\Delta_{\mathbf{R}}$ is $g^{-1}V_1 g$-invariant and $d(v_1(t)g\Delta) = d(g\Delta) < \sigma$ for all $t \in \mathbf{R}$. Since $\sigma \leqslant 1$ (as in the proof of Proposition A.7) we see that $g\Delta_{\mathbf{R}} = W_1$ or W_2. This implies that $x = g\Gamma \in X \cap A(q, \rho)$, where $q = 1$ or 2 and ρ is the determinant of the complete subgroup of Λ containing $g\Delta$ and spanning the same subspace. By the assumption on X we now get that $X \subset A(q, \rho)$. By our choice of σ we then have $\sigma \leqslant \rho$. But this is a contradiction since $\rho \leqslant d(g\Delta) < \sigma$. Hence the sets as above are unbounded and thus the proof is complete.

As noted earlier Propositions A.12 and A.9 yield parts a) and b) of Proposition 7, which thus stands proved. We next note the following variation of Theorem A.8, first proved by Margulis [16], which was used in the proof of Proposition 9.

A.13. THEOREM. *Let $n \geqslant 2$ be fixed. Let $\{u_t\}$ be a unipotent one-parameter subgroup of $SL(n, \mathbf{R})$ and let $x \in SL(n, \mathbf{R})/SL(n, \mathbf{Z})$. Then there exists a compact subset K of $SL(n, \mathbf{R})/SL(n, \mathbf{Z})$ such that $\{t \geqslant 0 \mid u_t x \in K\}$ is an unbounded subset of* $\mathbf{R}$.

Proof. Let $g \in G$ such that $x = gSL(n, \mathbf{Z})$ and let $\Lambda = g\mathbf{Z}^n$. In view of Lemma A.2 a) there exists $\sigma > 0$ such that $d(\Delta) > \sigma$ for all subgroups Δ of Λ. Hence by Theorem A.1 there exists $\delta > 0$ such that for any $T \geqslant 0$ there exists a $s \geqslant T$ for which $\| u_s \xi \| \geqslant \delta$ for all $\xi \in \Lambda - \{0\}$. Let $K = \{hSL(n, \mathbf{Z}) \mid \| hp \| \geqslant \delta$ for all $p \in \mathbf{Z}^n - \{0\}\}$. Then by the Mahler criterion, recalled earlier, K is a compact subset of $SL(n, \mathbf{R})/SL(n, \mathbf{Z})$. From the choices it is clear that $\{s \geqslant 0 \mid u_s x \in K\}$ is an unbounded subset. This proves the theorem.

REFERENCES

[1] Borel A. and G. Prasad. Valeurs de formes quadratiques aux points entiers. *C. R. Acad. Sci. Paris, Série I, 307* (1988), 217-220.

[2] Cassels, J.W.S. *An Introduction to the Geometry of Numbers.* Springer-Verlag, Berlin-Heidelberg, 1959.

[3] Dani, S.G. On invariant measures, minimal sets and a lemma of Margulis. *Invent. Math. 51* (1979), 239-260.

[4] —— Dynamics of flows on homogeneous spaces: A survey. *Proceedings of a conference at Guanajuato,* Mexico (1983), Aportaciones Matematicas 1, Soc. Mat. Mexicana, 1985.

[5] —— On orbits of unipotent flows on homogeneous spaces, II. *Ergod. Th. and Dynam. Syst. 6* (1986), 167-182.

[6] Dani, S.G. and G.A. Margulis. Values of quadratic forms at primitive integral points. *C. R. Acad. Sci. Paris, Série I, 308* (1989), 199-203.

[7] Dani, S.G. and G.A. Margulis. Values of quadratic forms at primitive integral points. *Invent. Math. 98* (1989), 405-424.

[8] Dani, S.G. and G.A. Margulis. Orbit closures of generic unipotent flows on homogeneous spaces of $SL/(3, \mathbf{R})$. *Math. Ann. 286* (1990), 101-128.

[9] Gaal, S.Å. *Linear Analysis and Representation Theory.* Springer-Verlag, Berlin-Heidelberg - New York, 1973.

[10] Goto, M. *An Introduction to Topological Groups.* Lecture Notes Series of Aarhus Univ., Mat. Inst., No. 40, 1974.

[11] Jacobson, N. *Lectures in Abstract Algebra, Vol. II — Linear Algebra.* D. Van Nostrand Co., Princeton (1953).

[12] Lang, S. *Calculus of Several Variables.* Addison-Wessley Publishing Co., 1973.

[13] Lekkerkerker, C.G. *Geometry of Numbers.* Wolters-Noordhoff-Publ. - Groningen, North. Holland Publ. Co., Amsterdam - London, 1969.

[14] Lewis, D.J. The distribution of values of real quadratic forms at integer points. *Proceed. of Symp. in Pure Math. Vol. XXIV,* Amer. Math. Soc., Providence, RI, 1973.

[15] Mahler, K. On lattice points in n-dimensional star bodies. *Proc. Royal Soc. Lond. A 187* (1946), 151-187.