ON THE INVERSIVE DIFFERENTIAL GEOMETRY OF PLANE CURVES

L'Enseignement Mathématique, t. 36 (1990), p. 175-196

ON THE INVERSIVE DIFFERENTIAL GEOMETRY OF PLANE CURVES

by G. CAIRNS [1]) and R. W. SHARPE [2])

§ 1. INTRODUCTION

In this article we study the extrinsic inversive geometry of curves in the Euclidean plane $\mathbf{R}^2$ under the inversive group $G = PSL_2(\mathbf{C})^{\sim}$ of *general* Möbius transformations. This is $PSL_2(\mathbf{C})$ extended by complex conjugation. $PSL_2(\mathbf{C})$ itself is the *special*, or orientation preserving Möbius transformations. An introduction to classical inversive geometry may be found in [18].

As our model for this geometry we take the complex plane $\mathbf{C}$ (with coordinate $z = x + iy$) together with the point at infinity, ∞. The underlying topological space is of course S^2 and G is the group of conformal and anticonformal transformations of S^2, but we use the standard Euclidean metric on $\mathbf{C}$. We shall assume that all our curves are oriented and smooth.

In § 2 we recall Coxeter's invariant (cf. [5]), the "inversive distance", between two non-intersecting circles. This is the imaginary part of their imaginary angle of intersection. Based on this idea we obtain a proof of a result of Kneser (cf. [9], p. 48) which says that on a vertex-free part of a curve γ the osculating circles never intersect. Using the square root of the inversive distance between neighbouring osculating circles on γ we obtain an invariant 1-form ω (the infinitesimal inversive arc-length). This 1-form was apparently first discovered by H. Liebmann in 1923 [12], although the name of G. Pick is also mentioned by Blaschke in [2]. If γ is parametrized by the arc-length s and if $\kappa(s)$ denotes the curvature at the point $\gamma(s)$, then the 1-form ω can be identified as the 1-form $\sqrt{|\kappa'(s)|}\, ds$ (cf. our § 2, or [3], p. 92), and can be extended continuously over the vertices. It follows that the set of vertices (points where $\kappa'(s) = 0$) of a curve is invariant under the inversive group. The integral of this invariant 1-form gives the inversive arc-length, $\nu = \int \omega$, a

---

[1]) This work of this author was done in part when he was a postdoctoral fellow at the University of Waterloo.

[2]) The work of this author was supported in part by National Science and Engineering Research Council of Canada grant A4621.

natural invariant parameter for curves in inversive geometry. We end the section with a table for the inversive arc-length for various conics.

The classical four vertex theorem, due to Mukhopadhaya in 1909, states that every simple closed curve in $\mathbf{R}^2$ has at least four vertices. Though the standard proof is easy in the case of convex curves, Kneser's 1911 [11] generalization to the non-convex case is strangely more complicated, and the result is usually stated without proof in introductory texts. Simple and elegant proofs have been given by Valette in 1957 [17] (cf. also Pinkall 1987 [15]) and Osserman in 1985 [14]. The theorem is also known to be true for S^2 but the usual proof is again quite complicated. Furthermore it is easy to construct simple closed curves on the torus with only two vertices. In § 3 we present a simple new proof of the four vertex theorem for (not necessarily convex) simple closed curves on $\mathbf{R}^2$ based on the conformal invariance of the vertices. The moral is that the four vertex theorem is really a theorem in inversive differential geometry, where the larger symmetry group is a powerful aid. In § 4 we consider a generalization of the form ω to curves γ on an arbitrary Riemannian surface given by the formula:

$$\omega_\gamma = \sqrt{|\kappa_g'|}\, ds$$

where κ_g is the geodesic curvature of the curve on the surface. It turns out that this form is invariant under maps between surfaces which preserve the curves of constant geodesic curvature, the so-called "concircular maps". As a consequence of this we show in § 5 the following result.

THEOREM. *If* γ *is a smooth, null-homotopic, simple closed curve on a complete Riemannian surface* M *of constant curvature, then the geodesic curvature of* γ *has at least four local extrema.*

The remainder of the paper continues a general study of curves in the inversive plane. The method used throughout is the method of moving frames in one of its simpler incarnations, systematically developed by A. Tresse [16] called "the method of reduced equations". In fact the spirit here is much the same as the first part of É. Cartan's beautiful book [4].

In § 6 we show that for each non-vertex point p on a curve γ there is a *unique* orientation preserving Möbius transformation $g \in G$ such that $g^{-1}(p) = 0$ and the Taylor expansion for the curve $g^{-1}(\gamma)$ at the origin has the normal form

$$y = \pm \frac{x^3}{6} + Q \frac{x^5}{60} + O(x^6) \tag{1.1}$$

where $\pm = \operatorname{sgn}(\kappa')$. The denominator 60 (rather than the seemingly more natural $5! = 120$) represents a normalization of Q to simplify formula 1.3 below and the calculations for the loxodrome in §9. It is clear that Q is invariant under (special) Möbius transformations and so we call it *the inversive curvature of* γ *at* p. It can be calculated in terms of the Euclidean curvature $\kappa(s)$ and its derivatives with respect to Euclidean arc-length by means of the formula

$$(1.2) \qquad Q = \frac{4(\kappa''' - \kappa^2\kappa')\kappa' - 5\kappa''^2}{8\kappa'^3}$$

We note that although the sign of Q depends on the orientation of the plane, it is nevertheless independent of the orientation of the curve. The curvature Q corresponds to the invariant $b/2$ which Blaschke ([2], end of §21) obtains by a completely different (and roundabout) method).

The procedure described above gives rise to a *Frenet lift* $g:\gamma - \{\text{vertices}\} \to G$, which is a curve on the Lie group G parametrized by inversive arc-length. In §7 we show that parallel translation of the tangent vector $dg/dv \in T_g(G)$ back to the identity by g^{-1} yields the formula

$$(1.3) \qquad g^{-1}\frac{dg}{dv} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2}\operatorname{sgn}(\kappa')(Q-i) & 0 \end{pmatrix}$$

It follows that the curvature Q *determines* the vertex-free curve up to a Möbius transformation.

The curves with Q constant are especially interesting as they constitute the "lines and circles" of inversive geometry. These are studied in §9 and turn out to be what Blaschke [2] calls "loxodromes"; that is, they are the equi-angular spirals (Bernouli's *spira mirabilis*) and their inversive images. Loxodromes are the orbits of 1-parameter subgroups of loxodromic transformations.

In §10 we use a simple notion of *contact* to define and determine the complex of smooth, local "geometric" differential forms Λ^*_{geo} on a vertex free curve in $\mathbf{R}^1$. This is a universal complex equipped with a homomorphism $\Psi_\gamma\colon \Lambda^*_{geo} \to \Lambda^*(\gamma)$ to the de Rham complex of γ for every vertex free curve γ, and satisfying the invariance property that $\Psi_\gamma = g^*\Psi_{g(\gamma)}$ for every $g \in G$. It turns out that Λ^*_{geo} is generated by the function Q and the form ω so that these are essentially the *only* interesting smooth local invariants of curves in $\mathbf{R}^2$.

§ 2. THE INFINITESIMAL COXETER INVARIANT

In his paper [5] Coxeter describes the "inversive distance" between two non-intersecting circles (cf. H. G. Forder [8] and H. W. Alexander [1] for alternate treatments). Starting with the standard formula

$$d^2 = a^2 + b^2 - 2ab\cos\theta$$

relating the sides and angle of a Euclidean triangle one obtains the formula:

$$\cos\theta = \frac{a^2 + b^2 - d^2}{2ab} \tag{2.1}$$

for the cosine of the angle between two intersecting circles in terms of the two radii a and b, and the distance d between the centres. Although the left hand side of 2.1 makes sense only for intersecting (or tangent) circles, the right hand side makes sense even for disjoint circles. The various cases are:

$\mu = \dfrac{a^2 + b^2 - d^2}{2ab}$	circles
$\lvert\mu\rvert < 1$	intersecting
$\mu = 1$	internal tangency
$\mu = -1$	external tangency
$\lvert\mu\rvert > 1$	disjoint

GEOMETRIC SIGNIFICANCE OF μ

Coxeter defines the *inversive distance* δ between *disjoint* circles by the formula $\cosh\delta = \lvert\mu\rvert$. Like the ordinary angle between two intersecting circles, inversive distance is a conformal invariant of the relative position of the two disjoint circles.

We shall apply this formula to compute the inversive distance between the osculating circles of two nearby points on a curve. Let γ be a smooth curve in $\mathbf{C}$ parametrized by arc-length $z : (\alpha, \beta) \to \mathbf{C}$. Let $\kappa(s)$ be the ordinary Euclidean curvature of γ at the point $z(s)$. The radius of the osculating circle is $1/\lvert\kappa\rvert$ and its centre lies at $z + iz'/\kappa$. Comparing the osculating circles at points $z(s)$ and $z(s+h)$ we find that the inversive distance between them is given by

$$\cosh \delta(s+h, s) = \frac{\dfrac{1}{\kappa(s+h)^2} + \dfrac{1}{\kappa(s)^2} - \left| z(s+h) - z(s) + i \left\{ \dfrac{z'(s+h)}{\kappa(s+h)^2} - \dfrac{z'(s)}{\kappa(s)^2} \right\} \right|}{\dfrac{2}{\kappa(s+h)\,\kappa(s)}}$$

Expanding this in a Taylor series in h gives:

$$(2.2) \qquad 1 + \frac{\delta^2}{2} + \ldots = \cosh \delta = 1 + \frac{1}{4!}\, \kappa'(s)^2 h^4 + O(h^5) \ .$$

We note in particular that if $\kappa'(s) \neq 0$, the right hand side of this expression is larger than 1 for small h, proving that the osculating circles of nearby points on a curve with $\kappa' \neq 0$ are disjoint. It follows from this that the set of all osculating circles of such a curve forms a nested family.

From 2.2 we get

$$\sqrt{\delta(s+h, s)} = 12^{-1/4} \sqrt{|\kappa'(s)|}\ h + O(h^2) \ .$$

It follows immediately that $\omega_\gamma = \sqrt{|\kappa'(s)|}\, ds$ is a differential 1-form on the curve which is invariant under the action of the Möbius group in the following sense: for all inversions φ, the 1-form ω_γ is the pull-back by φ of the 1-form $\omega_{\varphi(\gamma)}$ associated to the curve $\varphi(\gamma)$. We call $\omega = \omega_\gamma$ the "infinitesimal Coxeter invariant".

Definition. A vertex of γ is a zero of ω.

The invariance of ω means in particular that the property of being a vertex is an inversive invariant.

We define the inversive arc-length of a curve γ to be the integral

$$\int_\gamma \omega \ .$$

This is an inversive invariant of γ. Indeed, fixing a point $a \in \gamma$, we can parametrize the curve by means of the natural parameter ν, where

$$\nu(p) = \int_a^p \omega \ .$$

For example, the inversive arc-lengths for the conics can be calculated by means of the integrals

$$4\sqrt{3a(a^2-1)} \int_0^{\pi/2} \frac{\sqrt{\cos\theta\sin\theta}\, d\theta}{\cos^2\theta + a^2\sin^2\theta} \quad \text{for the ellipse} \quad x^2 + \frac{y^2}{a^2} = 1$$

and

$$4\sqrt{3a(a^2+1)} \cdot \int_0^{\pi/2} \frac{\sqrt{\cos\theta}\ d\theta}{a^2 + \cos^2\theta} \quad \text{for the hyperbola} \quad x^2 + \frac{y^2}{a^2} = 1\ .$$

Only in the case of a circle and a parabola do we get an elementary integral. Here is a table of the inversive arc-lengths of various conics.

Type	e	Inversive Length	Type	θ	Inversive Length
Circle	0.0	0.0	Parabola	0.0	$\sqrt{6}\ \pi = 7.6953$
Ellipses	0.1	0.59	Hyperbola	0.1 π	12.10
	0.2	1.19		0.2 π	10.71
	0.3	1.80		0.3 π	9.60
	0.4	2.45		0.4 π	8.63
	0.5	3.15		0.5 π	7.70
	0.6	3.92		0.6 π	6.77
	0.7	4.81		0.7 π	5.79
	0.8	5.91		0.8 π	4.68
	0.9	7.48		0.9 π	3.30

e = eccentricity $\qquad$ θ = angle between the asymptotes

TABLE 2.3

As the reader may show, when the eccentricity approaches 1 the inversive lengths of both the ellipses and the hyperbolas approach a common value which is *twice* the inversive length of the parabola. This fact and the above table are reminders that inversive geometry behaves quite differently than projective or affine geometry, which don't distinguish among the various ellipses for example. In particular it follows that the inversive arc-length is neither a projective nor an affine invariant.

We note also that the table gives the inversive lengths of any curve equivalent to these conics under a linear fractional transformation. For instance, inverting the parabola with respect to its focal point yields the cardioid which must therefore have the same inversive length as the parabola.

§3. THE FOUR VERTEX THEOREM IN $\mathbf{R}^2$

Let γ be a closed embedded curve on $\mathbf{R}^2$. The Euclidean curvature κ is defined and so it must have a minimum and a maximum which give two vertices on γ. (Indeed the number of local minima must be the same as the number of local maxima, so that the number of extrema is even.) Next we move γ by a Möbius transformation so as to send one of these extrema to ∞, and in such a way that the curve becomes asymptotic to the x-axis. Now $\kappa(s) \to 0$ as $s \to \pm\infty$, and the theorem of turning tangents ([6], p. 37) says that $\int \kappa(s)\, ds = 0$. It follows that κ cannot have just one maximum or just one minimum for if so it would have a fixed sign and then the integral could not be zero. Thus γ has at least 2 extrema in addition to the one at infinity. But since the total number of extrema is even, there must be at least four of them, and hence four vertices.

There is a subtle point which we have glossed over in this argument. The vertices come in two types. As well as the extrema of κ (the "honest" vertices) there may also be non-extremal critical points of κ. The above "proof" has used the fact that not only are the vertices inversive invariants, but so too are the isolated extrema. Whereas this is indeed true (as is implied by equation 4.3 of the next section), it suffices to note that the non-extremal critical points of κ are unstable phenomena and each of them may be eliminated by a deformation of the curve with support in a small neighborhood of it. One may thus assume that all of the vertices of γ are extrema, whereupon the above proof stands as is.

Remark 3.1. The reader may compare the above proof to that of [10], where the four vertex theorem is obtained by using a Möbius transformation to send a *non*-vertex point to infinity.

§4. A GENERALIZATION OF THE INVARIANCE OF ω

Let (M, h) be a Riemannian surface with metric h, and let γ be a curve on M with geodesic curvature κ_g. We can ask whether the 1-form along a curve γ given by

$$\omega_\gamma = \sqrt{|\kappa_g'|}\, ds$$

is a conformal invariant. More precisely, let $\Psi:(M_1, h_1) \to (M_2, h_2)$ be a conformal map and let γ_1 be a curve on M_1; is it true that

$$\Psi^*(\omega_{\Psi(\gamma_1)}) = \omega_{\gamma_1}\ . \tag{4.1}$$

A necessary condition for this is that Ψ send curves of constant geodesic curvature to curves of constant geodesic curvature. We show that this condition is in fact sufficient.

THEOREM 4.2. *Let* $\Psi : (M_1, h_1) \to (M_2, h_2)$ *be a conformal map which sends curves of constant geodesic curvature to curves of constant geodesic curvature. Then 4.1 holds for all curves in* M_1.

Proof. Let γ_1 be a curve on M_1 parametrized by arc-length s_1 and with geodesic curvature $\kappa_1(s_1)$; similarly let $\gamma_2 = \Psi(\gamma_1)$ be the image curve on M_2 parametrized by arc-length s_2 with geodesic curvature $\kappa_2(s_2)$. We must show that

$$\Psi^*\left(\sqrt{|\kappa_2'(s_2)|}\, ds_2\right) = \sqrt{|\kappa_1'(s_1)|}\, ds_1 \, .$$

We will prove this for orientation reversing conformal maps Ψ; the general case then follows by composition. We start with a technical lemma.

LEMMA. *If* $p \in \gamma_1$ *then there exists a neighborhood* Ω *of* x *in* M_1 *and a positively oriented orthonormal frame* $(\tau, \mathbf{n})$ *of vector fields on* Ω *such that the following properties hold.*

a) *The connected component containing* x *of the intersection of* γ_1 *with* Ω *is a flow line for* τ.

b) *The flow lines of* $\mathbf{n}$ *have constant curvature.*

Proof of Lemma. It suffices to take a sufficiently small neighborhood Ω of p such that there exist on it a flow $\mathbf{n}$ by geodesics which are perpendicular to the connected component of $\gamma_1 \cap \Omega$ containing x. Then τ is chosen perpendicular to $\mathbf{n}$.

Returning to the proof of the theorem, it obviously suffices to work on the neighborhood Ω of some arbitrary point x of M_1. Since Ψ is conformal there exists a positive function f on Ω such that $(\Psi_*(f\tau), -\Psi_*(f\mathbf{n}))$ is a positively oriented orthonormal framing of $\Psi(\Omega)$, where Ψ_* is the push-forward map on tangent vectors induced by Ψ. We get $g_1 = f^2\Psi^*(g_2)$, where $\Psi^*(g_2)$ is the pull-back of g_2 by Ψ. Now

$$\Psi^*\left(\sqrt{|\kappa_2'(s_2)|}\, ds_2\right) = f^{-1}\Psi^*\left(\sqrt{|\kappa_2'(s_2)}\right) ds_1$$

so it suffices to show that

$$\Psi^*(\kappa_2'(s_2)) = -f^2\kappa_1'(s_1) \, . \tag{4.3}$$

By the lemma we can take $\tau = d/ds_1$ on γ_1 so that

$$\kappa_1'(s_1) = \tau(\kappa_1), \quad \text{and}$$
$$\Psi^*\big(\kappa_2'(s_2)\big) = \Psi^*\big(\Psi_*(f\tau)\kappa_2\big) = f\tau\big(\Psi^*(\kappa_2)\big)$$

and hence it suffices to show that

$$\tau\big(\Psi^*(\kappa_2)\big) = -f\tau(\kappa_1) \tag{4.4}$$

Now the curvature κ_1 is given by the standard formula $\kappa_1 = -g_1(\nabla_\tau \mathbf{n}, \tau) = g_1([\mathbf{n}, \tau], \tau)$, where ∇ is the Levi-Civita connection and $[\mathbf{n}, \tau]$ is the Lie bracket of vector fields $\mathbf{n}$ and τ. Similarly the curvature κ_2 is given by

$$\kappa_2 = g_2\big([-\Psi_*(f\mathbf{n}), \Psi_*(f\tau)], \Psi_*(f\tau)\big) = g_2\big(\Psi_*[f\tau, f\mathbf{n}], \Psi_*(f\tau)\big)$$

and therefore

$$\begin{aligned} \Psi^*(\kappa_2) &= f^{-2}g_1\big([f\tau, f\tau\big) \\ &= f^{-2}g_1\big(f^2[\tau, \mathbf{n}] + f\tau(f)\mathbf{n} - f\mathbf{n}(f)\tau, f\tau\big) \\ &= -f\kappa_1 - \mathbf{n}(f) \end{aligned} \tag{4.5}$$

Thus, in order to prove 4.4, we will show that

$$\begin{aligned} \tau\big(f\kappa_1 + \mathbf{n}(f)\big) &= f\tau(\kappa_1) \\ \text{i.e.} \quad \kappa_1\tau(f) + \tau\big(\mathbf{n}(f)\big) &= 0\,. \end{aligned} \tag{4.6}$$

To do this write $[\mathbf{n}, \tau]$ as a linear combination of τ and $\mathbf{n}$

$$[\mathbf{n}, \tau] = \kappa_1\tau + \mu_1\mathbf{n} \tag{4.7}$$

where of course μ_1 is the geodesic curvature of the flow lines of $\mathbf{n}$. If μ_2 is the geodesic curvature of the flow lines of $\Psi_*(\mathbf{n})$, then analogously to 4.5 we have

$$\Psi^*(\mu_2) = -f\mu_1 + \tau(f)\,. \tag{4.8}$$

Since the flow lines of $\mathbf{n}$ have constant curvature we have

$$\mathbf{n}(\mu_1) = 0 \tag{4.9}$$

Since Ψ sends curves of constant curvature to curves of constant curvature we also have

$$\mathbf{n}\big(\Psi^*(\mu_2)\big) = \Psi^*\big(\Psi_*(\mathbf{n})\mu_2\big) = 0 \tag{4.10}$$

Applying $\mathbf{n}$ to 4.8 yields, in view of 4.9) and 4.10)

$$\mathbf{n}(f)\mu_1 = \mathbf{n}\tau(f)$$

Combining this with 4.5 yields

$$\mathbf{n}(\tau(f)) - \tau(\mathbf{n}(f)) = [\mathbf{n}, \tau](f) = \kappa_1\tau(f) + \mu_1\mathbf{n}(f) = \kappa_1\tau(f) + \mathbf{n}\tau(f)$$

which gives 4.6 as required.

§5. A generalized four vertex theorem

The curves of constant curvature in the round 2-sphere S^2, the upper half plane H^2 (hyperbolic space), and the Euclidean plane $\mathbf{R}^2$ are just the circles. Moreover, the stereographic projection $p: S^2 \to \mathbf{R}^2$ and the inclusion $i: H^2 \to \mathbf{R}^2$ both preserve these circles. Thus theorem 4.2 says that our form

$$\omega = \sqrt{|\kappa'(s)|}\, ds$$

along a curve γ in $\mathbf{R}^2$ pulls back via p or i to the form

$$\omega = \sqrt{|\kappa_g'(s)|}\, ds$$

along the corresponding curve γ', where here $\kappa_g(s)$ and s refer to the geodesic curvature and arc-length of γ' in the metric for S^2 or H^2. Thus we obtain the four vertex theorem for S^2 and H^2. It follows that the four vertex theorem holds for all complete simply connected Riemannian surfaces of constant curvature. Finally if γ is a null-homotopic smooth simple closed curve on an arbitrary complete Riemannian surface M of constant curvature, then γ lifts one-to-one to a smooth simple closed curve with the same number of vertices on the simply connected universal cover of M. Once again it follows that the number of vertices is at least four.

Remark 5.1. Interestingly, simple closed homotopically non-trivial curves in the real projective plane always have at least three vertices [17]. Note that in non-orientable surfaces the number of honest vertices of a closed curve need not necessarily be even, since here geodesic curvature is only defined up to a sign.

§6. Normal form and inversive curvature

Let p be a non-vertex point of an oriented curve γ. Since the subgroup of Euclidean motions in G acts transitively on the points of $\mathbf{R}^2$ and the unit tangent vectors at these points, we may assume that the point $p \in \gamma$ which interests us lies at the origin with tangent vector in the positive x-direction. Then the curve has the following Taylor series at the origin

$$y = Ax^2 + Bx^3 + Cx^4 + Dx^5 + O(x^6) \ . \tag{6.1}$$

The coefficients in 6.1 can be expressed in terms of the Euclidean curvature at $p \in \gamma$ and its derivatives with respect to Euclidean arc-length according to the following formulas.

$$A = \frac{1}{2}\,\kappa$$

$$B = \frac{1}{6}\,\kappa'$$

$$C = \frac{1}{24}\,(\kappa'' + 3\kappa^3)$$

$$D = \frac{1}{120}\,(\kappa''' + 19\kappa^2\kappa') \ .$$

Next we use the non-Euclidean motions of G to further normalize the equation for γ at p. Writing

$$F(z) = -y + Ax^2 + Bx^3 + \ldots$$

we have

$$\gamma = \{z \in \mathbf{C} \,|\, F(z) = 0\}, \text{ so that } g^{-1}(\gamma) = \{g^{-1}(z) \,|\, F(z) = 0\}$$
$$= \{z \,|\, F(g(z)) = 0\} \ .$$

It is rather tedious to calculate the following sequence of transformations of the equations and so we suppress the algebraic work. Since we assume that p is not a vertex we have $B = \kappa'/6 \neq 0$. The substitution

$$z \mapsto \frac{z}{1 - iAz}$$

replaces 6.1 by a new series, which when solved for y yields the Taylor series

$$y = Bx^3 + (C - A^3)x^4 + (D - 4A^2B)x^5 + O(x^6) \ . \tag{6.2}$$

Next the substitution

$$z \mapsto \frac{z}{1 + \dfrac{C - A^3}{B}\,z}$$

applied to 6.2 yields, after solving for y,

$$y = Bx^3 + \left(D - 4A^2B - \frac{(C-A^3)^2}{B}\right) x^5 + O(x^6) \tag{6.3}$$

Finally the substitution

$$z \mapsto \frac{1}{\sqrt{|6B|}}\, z$$

applied to 6.3 yields

$$y = \pm\frac{x^3}{6} + Q\frac{x^5}{60} + O(x^6) \tag{6.4}$$

where $\pm$ is the sign of κ', and

$$Q = \frac{5}{3}\left\{\frac{D - 4A^2B}{B^2} - \frac{(C-A^3)^2}{B^3}\right\}$$

Expressing Q in terms of the Euclidean curvature we have

$$Q = \frac{4(\kappa''' - \kappa^2\kappa')\kappa' - 5\kappa''^2}{8\kappa'^3}$$

In particular our calculation shows that there exists an orientation preserving group element g^{-1} moving an arbitrary oriented non-vertex point to the origin in such a way that the image curve has for its Taylor expansion the normal form

$$y = \pm\frac{x^3}{6} + O(x^5)$$

and is oriented in the positive x-direction. The uniqueness of g^{-1} follows by showing the stabilizer of this normal form to be the identity. We omit this calculation. It follows that Q, *the inversive curvature of* γ, is invariant under the group of orientation preserving Möbius transformations.

At a vertex things work out somewhat differently. Assuming that a vertex is *non-degenerate,* in the sense that $\kappa'' \neq 0$ there, one finds, on attempting to imitate the above reduction, that the fourth order term cannot now be eliminated whereas the fifth order term can be removed. One finds that the normal form at a non-degenerate vertex is

$$y = \frac{1}{24}x^4 + \left\{\frac{\kappa''''}{720} - \frac{7}{1440}\kappa^2\kappa'' - \frac{1}{800}\frac{\kappa'''^2}{\kappa''}\right\}\kappa''^{-5/3}x^6 + O(x^7)\,.$$

The stabilizer of the expression $y = x^4/24 + O(x^6)$ has order 2 and is generated by

$$z \mapsto -\bar{z}\,.$$

It follows that the non-degeneracy of a vertex is an invariant of inversive geometry.

§7. THE CANONICAL MAP $g:\gamma \to G$

The considerations of the last section allow us to define a canonical map $g_\gamma:\gamma \to G$ for vertex free curves γ by mapping a point $p \in \gamma$ to $g_\gamma(p) \in G$, which is the unique group element such that $g_\gamma(p)^{-1}$ sends p to the origin and $g_\gamma(p)^{-1}(\gamma)$ has oriented contact of order 4 with the standard curve $y = x^3/6$ at the origin. We note that if $\gamma' = h(\gamma)$ for some $h \in G$, then obviously $g_{\gamma'}(h(p)) = h(g_\gamma(p))$. Of course altering the initial choice of the origin and the axes used there to describe the model will alter g_γ, but only by right multiplication by some fixed element of G. If $\sigma:(\alpha,\beta) \to \mathbf{C}$ is a parametrization of the curve by Euclidean arc-length s, and $\sigma'(s) = e^{i\theta(s)}$, then the curvature of the curve at $\sigma(s)$ is $\theta'(s) = \kappa(s)$, and we have the following explicit formula for g.

$$g(s) = \begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (\kappa''-2i\kappa\kappa')/4\kappa' & 1 \end{pmatrix} \begin{pmatrix} |\kappa'|^{-1/4} & 0 \\ 0 & |\kappa'|^{1/4} \end{pmatrix}$$

The first two factors are Euclidean motions whose inverse puts γ into oriented first order contact with the oriented x-axis. The rest improve the order of contact to 4 as in §6. It is convenient to regard g as a function of the inverse arc-length ν. Now $g(\nu)$ is a curve on the Lie group G, with tangent vector $dg/d\nu$ at $g(\nu)$. Left translation by $g(\nu)^{-1}$ moves this tangent vector to the origin to yield

$$c(\nu) = g(\nu)^{-1}\frac{dg}{d\nu} \tag{7.1}$$

which is a vector in the Lie algebra $sl_2(\mathbf{C})$ of 2 by 2 complex matrices of trace zero. As ν varies $c(\nu)$ inscribes a curve on this Lie algebra. Indeed it is well known (e.g. [13], p. 71) that this curve determines the original curve $g(\nu)$ up to left translation by an arbitrary constant element of G. Here is an explicit formula for the curve $c(\nu)$. It is easy but rather tedious to verify it.

$$c(\nu) = \begin{pmatrix} 0 & 1 \\ T & 0 \end{pmatrix}, \quad \text{where} \quad T = \frac{1}{2}\operatorname{sgn}(\kappa')\,(Q-i)$$

and Q is as in § 6. It follows that the inversive curvature Q determines the curve up to an orientation preserving inversive automorphism.

§ 8. Relation with Cartan's moving frames

Let us sketch a more usual way of obtaining a Frenet lift. The connection with the Schwartzian described here can be found, for example, in Cartan's book [4] and very succintly in [7]. The canonical line bundle

$$p : \xi \to \mathbf{P}^1(\mathbf{C})$$

has a pedestrian description (away from the zero-section) as:

$$\begin{array}{ccc} (z_1, z_2) \in \xi - \{\text{zero section}\} & = & \mathbf{C}^2 - \{0\} \\ \updownarrow & & \downarrow p \\ z = \dfrac{z_1}{z_2} \leftrightarrow [z_1, z_2] \in \mathbf{P}^1(\mathbf{C}) & & \end{array}$$

Let $\sigma : (\alpha, \beta) \to \mathbf{R}^2 \subset \mathbf{P}^1(\mathbf{C})$ be a curve; we choose an arbitrary lift $\hat{\sigma} = (z_1(t), z_2(t))$ and set $f_1 = \lambda\hat{\sigma}, f_2 = \dot{f}_1 = \dot{\lambda}(z_1, z_2) + \lambda(\dot{z}_1, \dot{z}_2)$, where $\cdot = \dfrac{d}{dt}$. Thus (f_1, f_2) is a frame in $\mathbf{C}^2$. We try to choose λ so that this frame has area 1. The condition on λ is:

$$\begin{aligned} 1 = \operatorname{Area}(f_1, f_2) &= \operatorname{Area}\big(\lambda(z_1(t), z_2(t)),\ \lambda(\dot{z}_1, \dot{z}_2)\big) \\ &= \lambda^2(z_1\dot{z}_2 - z_2\dot{z}_1), \text{ or } 1 = -(\lambda z_2)^2 \dot{z}\,. \end{aligned}$$

Thus $\lambda = \dfrac{i}{z_2\sqrt{\dot{z}}}$ will do, and we have

$$f_1 = \frac{i}{\sqrt{\dot{z}}}\,(z, 1)\,,$$

$$\text{and} \quad f_2 = \dot{f}_1 = -\frac{1}{2}\, i\,\ddot{z}\,\dot{z}^{-3/2}(z, 1) + i\,\dot{z}^{-1/2}(\dot{z}, 0)\,.$$

Finally a calculation shows that $\dot{f}_2 = Sf_1$, where $S = \dfrac{3}{4}\,\ddot{z}^2\dot{z}^{-2} - \dfrac{1}{2}\,\dddot{z}\,\dot{z}^{-1}$.

Of course S is the *Schwartzian derivative* which this calculation interprets as a "curvature" of σ. Now the Schwartzian S depends on the particular parametrization which is used for the curve. For our purposes we wish to use *inversive arc-length* as the parameter, so that the "curvature" S becomes an *intrinsic* invariant of the curve in inversive geometry. And in this case it turns out that S has constant imaginary part. To see this we describe S in terms of the more familiar Euclidean curvature and its derivatives with respect to Euclidean arc-length.

The Euclidean and inversive arc-lengths are related by the equation:

$$d\nu = |\kappa'|^{1/2} ds, \quad \text{where} \quad ' = \frac{d}{ds}. \text{ Thus:}$$

$$\dot{z} = z'|\kappa'|^{-1/2} = e^{i\theta}|\kappa'|^{-1/2},$$

$$\ddot{z} = \operatorname{sgn}(\kappa')e^{i\theta}\left\{-\frac{1}{2}\frac{\kappa''}{\kappa'^2} + i\frac{\kappa}{\kappa'}\right\},$$

$$\dddot{z} = \operatorname{sgn}(\kappa')e^{i\theta}|\kappa'|^{-1/2}\left\{-\frac{1}{2}\frac{\kappa'''}{\kappa'^2} + \frac{\kappa''^2}{\kappa'^3} - \frac{\kappa^2}{\kappa'} + i\left(1 - \frac{3}{2}\frac{\kappa\kappa''}{\kappa'^2}\right)\right\}$$

Using these expressions we can calculate the Schwartzian as:

$$\frac{3}{4}\left(\frac{\ddot{z}}{\dot{z}}\right)^2 - \frac{1}{2}\frac{\dddot{z}}{\dot{z}} = \operatorname{sgn}(\kappa')\left\{\frac{4(\kappa''' - \kappa^2\kappa')\kappa' - 5\kappa''^2}{16\kappa'^3} - \frac{i}{2}\right\} = \frac{1}{2}\operatorname{sgn}(\kappa')\,(Q - i)$$

Regarding the vectors f_1 and f_2 as column vectors, we obtain a 2 by 2 matrix $h = (f_2, f_1) \in G$, and according to the calculation above we have:

$$(\dot{f}_2, \dot{f}_1) = (f_2, f_1)\begin{pmatrix} 0 & 1 \\ S & 0 \end{pmatrix}$$

Thus $h(\nu)$ and $g(\nu)$ (cf. §7) are equal up to left multiplication by a constant element of G. This interprets Cartan's canonical frame (f_1, f_2) as the unique frame (up to a constant element of G) forming the columns of a matrix in G which moves the standard curve $y = x^3/6$ to the given curve with contact up to 4th order at the given point.

§9. Loxodromes

To calculate the curves with Q constant we solve the equation:

$$\frac{dg}{d\nu} = g\begin{pmatrix} 0 & 1 \\ \frac{1}{2}\varepsilon(Q - i) & 0 \end{pmatrix}, \quad \text{where } \varepsilon = \operatorname{sgn}(\kappa')$$

and use the fact that $\sigma(v) = g(v)\cdot 0$ to obtain the curve (where $\cdot$ refers to the action of $Sl_2(\mathbf{C})$ on $\mathbf{C}$ by Möbius transformations).

Now

$$\begin{pmatrix} 0 & 1 \\ \frac{1}{2}\varepsilon(Q-i) & 0 \end{pmatrix} = A\begin{pmatrix} \xi & 0 \\ 0 & -\xi \end{pmatrix} A^{-1},$$

where

$$\xi = \pm\frac{1}{\sqrt{2}}(1+Q^2)^{1/4}\sqrt{\varepsilon}\, e^{-\frac{1}{2}i\tan^{-1}\left(\frac{1}{Q}\right)}$$

and

$$A = \frac{1}{\sqrt{-2\xi}}\begin{pmatrix} 1 & 1 \\ \xi & -\xi \end{pmatrix}.$$

Thus $\dfrac{dgA}{dv} = gA\begin{pmatrix} \xi & 0 \\ 0 & -\xi \end{pmatrix}$ and hence $gA = C\begin{pmatrix} e^{\xi v} & 0 \\ 0 & e^{-\xi v} \end{pmatrix}$, where C is an invertible constant 2 by 2 matrix. Since $A\cdot -1 = 0$ we have

$$g\cdot 0 = -C\cdot(e^{2\xi v})$$

which is a linear fractional image of an equiangular spiral.

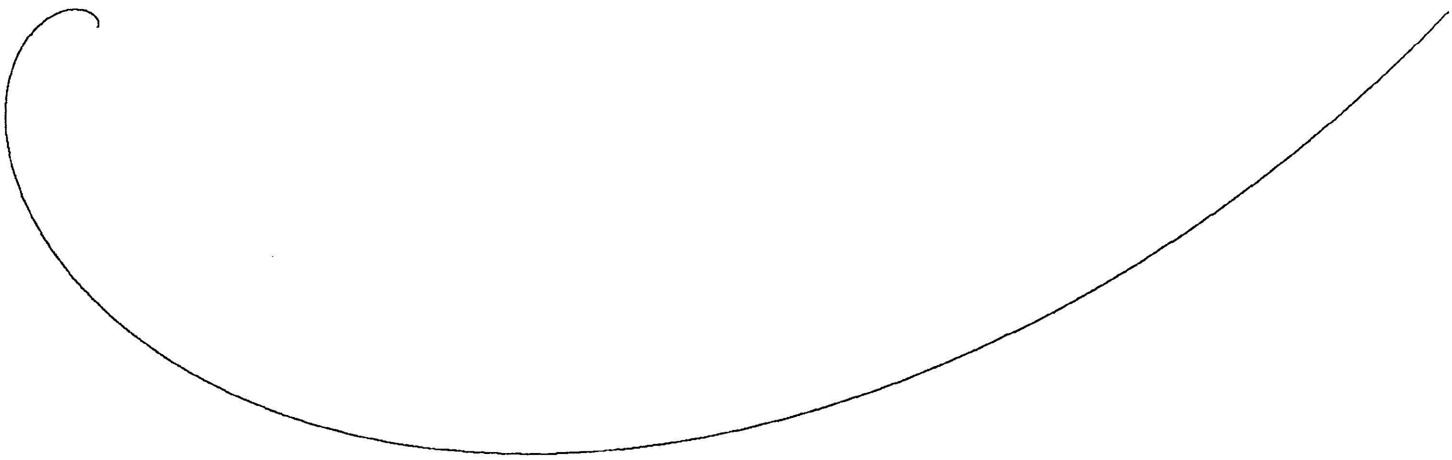

FIGURE 9.1

In particular, curves for which $Q = 0$, which we may call *inversive geodesics,* have

$$2\xi = \pm\sqrt{2\varepsilon}\, e^{-\frac{1}{2}\frac{\pi}{2}i} = \pm 1 \pm i$$

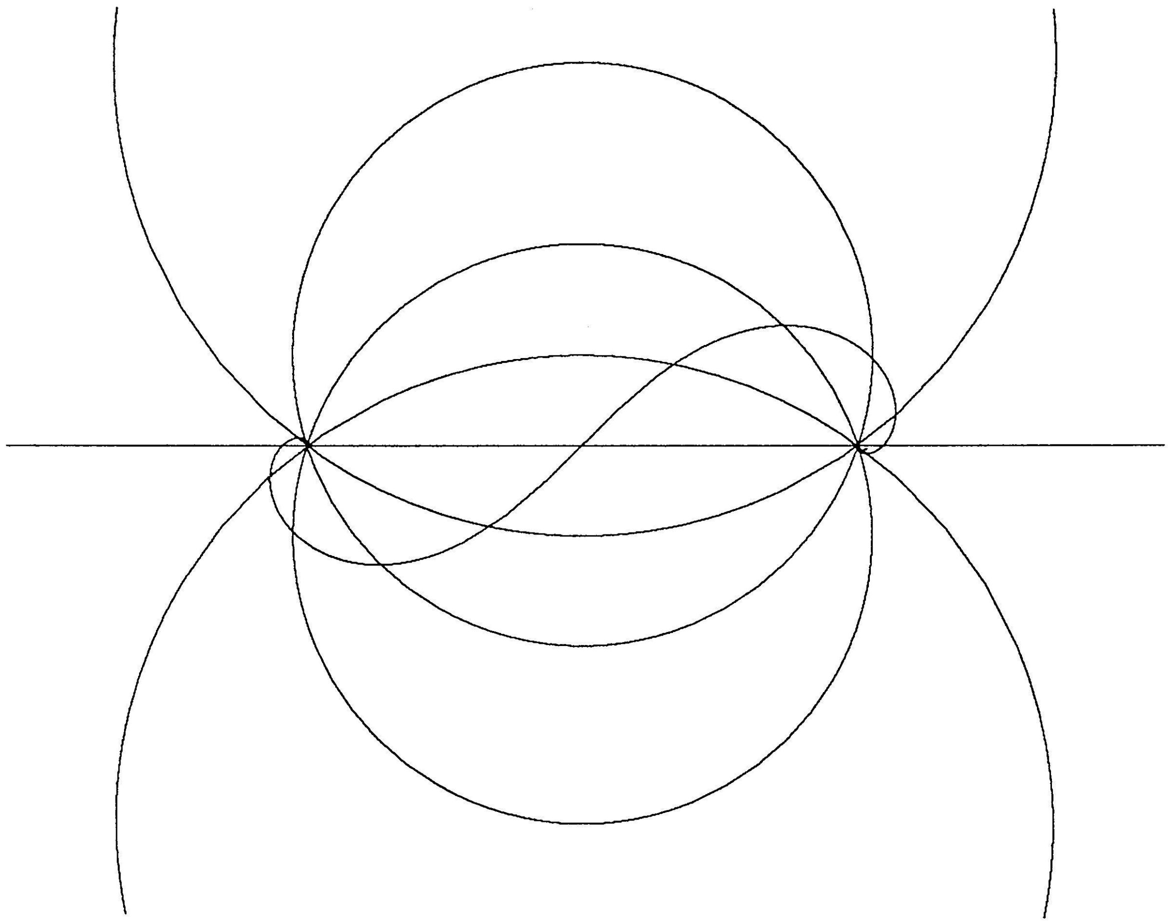

FIGURE 9.2

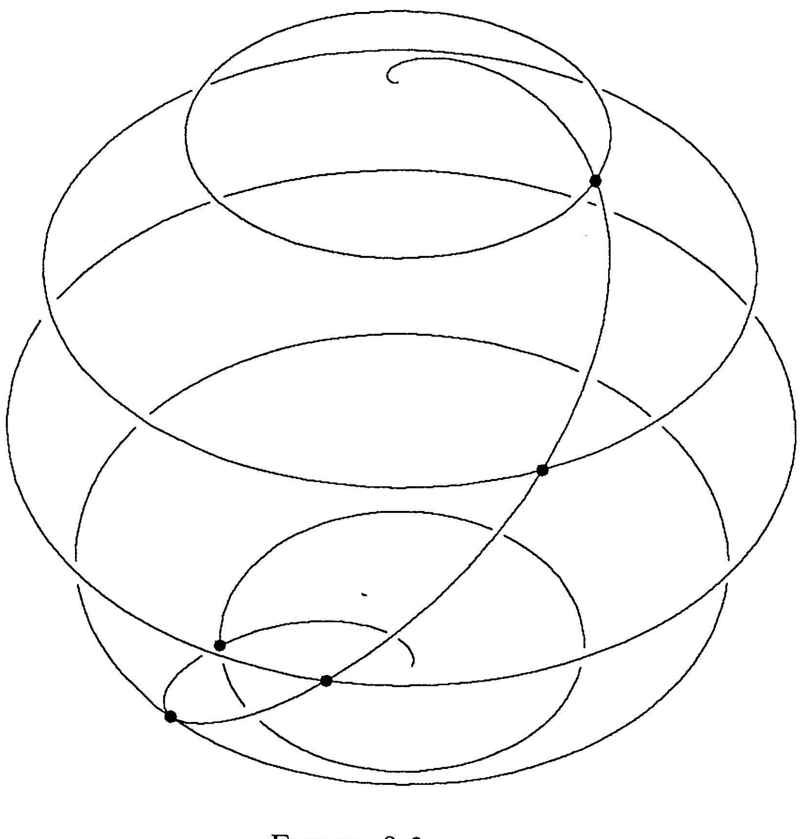

FIGURE 9.3

and are linear fractional images of the equiangular spiral with angle $\pm \pi/4$ given by

$$\sigma(v) = e^{(1 \pm i)v} = e^{v} e^{\pm iv} .$$

We note in particular that the inverse length of "one loop" of the inversive geodesic is 2π. Figure 9.1 is a picture of one such loop.

Equiangular spirals have two accumulation points, the poles, one at the origin and the other at infinity. These poles determine the family of circles through them (straight lines in this case) as well as a second family of circles orthogonal to the first. The equiangular spiral meets each family in fixed angles. The same is true for linear fractional images of this configuration, and with the same angles.

The connection between Q and the angle $\varphi \in (0, \pi/2)$ between the loxodrome and its first family of circles is given by

$$\tan \varphi = Q + \sqrt{Q^2 + 1} .$$

In figure 9.2, we show the inversive geodesic with poles at ± 1 together with its first family of circles.

In figure 9.3 we see the loxodrome again, in a perspective view this time, thrown up onto the two-sphere by the inverse of stereographic projection, along with its second family of circles.

We remark that it seems to be impossible to show the inverse geodesic in such a way as to allow more than one or two loops to appear to the eye, while at the same time allowing no distortion of the figure. This may account for a number of distorted diagrams of this loxodrome which have appeared in the literature. Of course one can picture many loops of some equiangular spirals, say with $Q \gg 0$. At the other extreme with $Q \ll 0$ we have a circumstance for which, in any scale, the corresponding equiangular spiral appears to the eye to be a straight line issuing from the origin. However as one "zooms" in or out this "straight line" appears to rotate about the origin.

§ 10. THE COMPLEX OF GEOMETRIC FORMS ON A CURVE IN $\mathbf{R}^2$

Among the various forms on a curve in $\mathbf{R}^2$, some, such as ω and Q, can be thought of as arising from the local way in which the curve is embedded in $\mathbf{R}^2$; that is they arise from the local geometric nature of the embedding and are invariant under Möbius transformations. These are the "smooth local geometric forms'' of inversive geometry, or ''geometric forms'' for short. To be more precise, suppose that $p_j \in \gamma_j (j=1,2)$, where at p_j the germ of γ_j has a local description of the form $\gamma_j = \{z \in R^2 | F_j(z) = 0\}$. We say (γ_1, p_1) and (γ_2, p_2) have contact of order at least r if for some choice of the F_j there is a Möbius transformation $g \in G$ moving p_1 to p_2 in such a way that the Taylor series in x and y for $F_2(x,y)$ and $F_1 \circ g(x,y)$ are equivalent in total degrees $\leqslant r$. We define a geometric form η of dimension d and order r to be an assignment $\gamma \mapsto \eta_\gamma$ which attaches to each vertex free curve γ a d-form η_γ on it such that the assignment satisfies:

i) *Invariance.* If $p_j \in \gamma_j (j=1,2)$ have contact of order at least r via the element $g \in G$, then

$$\eta_{\gamma_1}(p_1) = g^*(\eta_{\gamma_2}(p_2))$$

ii) *Smoothness.* If $\gamma(t_1, t_2, \dots t_k)$ is a smooth k-parameter family of curves with parametrization $\sigma(t, t_1, t_2, \dots t_k)$, then the function

$$\eta_{\gamma(t_1,t_2,\dots t_k)} \quad \text{if} \quad d=0, \quad \text{or} \quad \eta_{\gamma(t_1,t_2,\dots t_k)}\left(\frac{\partial\sigma}{\partial t}\right) \quad \text{if} \quad d=1$$

depends smoothly on $t, t_1, t_2, \dots t_k$.

The following lemma relates inversive curvature to contact.

LEMMA 10.1. *There are universal polynomials* $P_r(v_0, v_1, v_2, \dots v_{r-5}) \in \mathbf{Q}[v_0, v_1, v_2, \dots v_{r-5}]$ *for every* $r \geqslant 5$ *with the property that at any non-vertex point* p *on a curve* γ *with inversive curvature function* Q, *the pair* (γ, p) *has contact of order* $\geqslant r$ *with the curve*

$$y = \frac{x^3}{6} + t_5 x^5 + t_6 x^6 + \dots + t_r x^r \tag{10.2}$$

at the origin, where $t_k = P_k(Q, Q^{(1)}, Q^{(2)}, \dots Q^{(k-5)})(p)$ *for* $k = 5, 6, \dots, r$.

Proof. We regard the coefficients t_k as functions on the curve γ; that is, for each non-vertex point p on the curve, there is a unique inversive transformation sending p to the origin and throwing the curve into the form 10.2 (cf. §6) and so the coefficients $t_k(p)$ are uniquely determined by the curve γ and the point p. Thus the t_k's are functions on the curve; for example $t_5 = Q/60$ by the results of §6. By the formula 6.1 we see that $\kappa' = 1$ at the origin for the curve 10.2 which implies that

$$\frac{dt_k}{dv} = \frac{dt_k}{dx}$$

so that x can be used as inversive arc-length parameter to first order for the curve at the origin. Given the t_k's at the origin we can try to calculate them at a nearby point on the curve $(h, 0) + O(h^2)$. Displacing this point to the origin yields the following expression for the translated curve

$$y = \frac{(x+h)^3}{6} + \sum_{k=5}^{r} t_k(x+h)^k + O((x+h)^{r+1}) .$$

Let I_k be the ideal generated by $t_5, t_6, \dots t_{k-1}, x^k, h^2$, so that this equation implies

$$y = \frac{h}{2}x^2 + \frac{x^3}{6} + hkt_k x^{k-1} \mod I_k .$$

Then the substitution

$$z \mapsto \frac{z}{1 - i\frac{h}{2}z} = z + i\frac{h}{2}z^2 + O(h^2)$$

throws the equation into the form

$$y = \frac{x^3}{6} - \frac{5}{72}hx^6 + hkt_k x^{k-1} \mod I_k .$$

Since there is no quartic term mod I_k, this is already the normal form we seek, and we have shown that

$$t_{k-1}(h) = hkt_k(0) + A + Bh + O(h^2), \quad \text{where} \quad A, B \in \mathbf{Q}[t_5, t_6, \dots, t_{k-1}] .$$

Thus

$$\frac{dt_{k-1}}{dv} = kt_k + B$$

and hence

$$t_k \in \mathbf{Q}\left[t_5, t_6, \dots, t_{k-1}, \frac{dt_{k-1}}{dv}\right] .$$

Since $t_5 = Q/60$ it follows inductively that

$$t_k \in \mathbf{Q}[Q, Q^{(1)}, \dots, Q^{(k-5)}]$$

This completes the proof of the lemma.

Now we describe the universal construction for the geometric forms. Fix an infinite sequence of real variables $x_0, x_1, x_2, \dots$ and let A be the algebra of smooth real valued functions in these variables such that each function depends on only finitely many of them. We set:

$$\Lambda^d_{geo} = \begin{cases} A & \text{for} \quad d = 0, 1 \\ 0 & \text{otherwise} . \end{cases}$$

Then we define

$$d : \Lambda^0_{geo} \to \Lambda^1_{geo}$$

$$\text{by} \quad df = \sum_i x_{i+1} \frac{\partial f}{\partial x_i} .$$

Given a specific vertex free curve γ, let $\Lambda^*(\gamma)$ denote the DeRham complex. The map

$$\Psi_\gamma : \Lambda^d_{geo} \to \Lambda^d(\gamma)$$

is defined by:

$$f \mapsto f(Q^{(0)}, Q^{(1)}, \ldots) \qquad \text{for} \quad d = 0$$
$$f \mapsto f(Q^{(0)}, Q^{(1)}, \ldots)\omega \quad \text{for} \quad d = 1 .$$

This map is clearly a chain map. Moreover it is clear that for any form $\eta \in \Lambda^*_{geo}$, the assignment $\gamma \to \Psi_\gamma(\eta)$ is a geometric form. We claim that in fact *every* geometric form arises in this way. Since every geometric 1-form Ω is a multiple of the non-vanishing geometric 1-form ω, we may write $\Omega = R\omega$, where R is a geometric function. Thus it suffices to show that every geometric *function* H is of the form $\gamma \to \Psi_\gamma(\eta)$ for some function $\eta \in \Lambda^0_{geo}$. To see this we first consider the smooth $r - 4$ parameter family of curves P given by the equation

$$y = x^3 + t_5\, x^5 + t_6 x^6 + \ldots + t_r x^r .$$

Set $t = (t_5, t_6, \ldots t_r)$. These curves are all vertex free at the origin, and by the result of § 4 we know that for an arbitrary curve γ and an arbitrary point $p \in \gamma$ on it, (γ, p) has contact of order $\geq r$ with *some* member of this $r - 4$ parameter family of curves. It follows from the invariance property (i) that we need only find $\eta \in \Lambda^0_{geo}$, such that $\Psi_\gamma(\eta) = H_\gamma$ at the origin for all γ in the family P. By the smoothness property (ii) we can write $H_{\gamma(t)}(0) = L(t)$, for some smooth function L, and by the lemma above $t_5, t_6, \ldots t_r \in \mathbf{Q}[Q, Q^{(1)}, \ldots, Q^{(r)}]$. Thus $L(t) = \eta(Q^{(0)}, Q^{(1)}, \ldots, Q^{(r)})$ for some smooth function $\eta \in \Lambda^0_{geo}$, and we are done.

We remark that although Λ^*_{geo} gives all the smooth *local* invariants of curves in $\mathbf{R}^2$, it certainly does not give other, more global, invariants like $\nu = \int \omega$.

BIBLIOGRAPHY

[1] ALEXANDER, H. W. Vectorial inverse and non-Euclidean geometry. *Amer. Math. Monthly 74* (1967), 128-140.
[2] BLASCHKE, W. *Differentialgeometrie III.* Springer, Berlin, 1929.
[3] BUSHMANOVA, G. V. and NORDEN, A. P. *Elementy konformoi geometrii.* Izdatelstvo Kazanskogo Universiteta, 1972.
[4] CARTAN, E. *Leçons sur la théorie des espaces à connexion projective.* Cahiers Scientifiques fasc. 17, Gauthiers-Villars, Paris, 1937.
[5] COXETER, H. S. M. Inversive distance. *Ann. Mat. Pura Appl. (4) 71* (1966), 73-83.
[6] DOCARMO, M. *Differential Geometry of Curves and Surfaces.* Prentice Hall, Englewood Cliffs, 1976.
[7] FLANDERS, Harley. The Schwartzian as a curvature. *Journal of Differential Geometry 4* (1970), 515-519.
[8] FORDER, H. G. *The Calculus of Extension.* Chelsea, New York, 1960.
[9] GUGGENHEIMER, H. *Differential Geometry.* Dover, New York, 1977.
[10] HEIL, E. Some vertex theorems proved by means of Möbius transformations. *Ann. Mat. Pura Appl. 85* (1970), 301-306.
[11] KNESER, A. Bemerkungen über die Anzahl der Extreme der Krümmung auf geschlossenen Kurven und über verwandte Fragen in einer nichteuklidischen Geometrie. Festschrift zum 70. Geburtstag vo H. Weber 1912, 170-180.
[12] LIEBMANN, H. *Beiträge zur Inversionsgeometrie der Kurven.* Akad. der Wissenschaft, Munich 1923.
[13] MALLIAVIN, Paul. *Géométrie différentielle intrinsèque.* Hermann, Paris, 1972.
[14] OSSERMAN, R. The four or more vertex theorem. *Amer. Math. Monthly, 92* (1985), 332-337.
[15] PINKALL, U. On the four vertex theorem. *Aequationes Math. 34* (1987), 221-230.
[16] TRESSE, A. Sur les invariants différentiels des groupes continus de transformations. *Acta Math. 18* (1894), 1-95.
[17] VALETTE, G. Quelques propriétés conformes globales des courbes planes. *Acad. Roy. Belg. Bull. Cl. Sci. (5) 43* (1957), 66-79.
[18] WILKER, J. B. Inversive Geometry. In *The Geometric Vein,* ed. C. Davis, B. Grunbaum and F. A. Scherk, Springer-Verlag, New York, 1981.

(Reçu le 2 novembre 1989)

Grant Cairns

Dept. of Mathematics
La Trobe University
Bundoora VIC 3083
Australia

Richard W. Sharpe

Dept. of Mathematics
University of Toronto
Toronto M5S 1A1
Canada