2. Highest weight representations

all $k, l, m \in \mathbf{Z}_+$. If $u = u_1 u_2 \dots u_n$ is any monomial in the generators of degree n, define its index

$$\operatorname{ind}(u) = \sum_{i<j} \varepsilon_{ij}$$

where

$$\varepsilon_{ij} = \begin{cases} 0 & \text{if} \quad u_i \prec u_j \\ 1 & \text{if} \quad u_j \prec u_i \,. \end{cases}$$

Using Lemma 1.7, each monomial can be written as a sum of monomials of smaller degree, or smaller index, and hence, by an obvious induction, as a sum of monomials of index zero.

2. Highest weight representations

By analogy with the definition of highest weight representations of semi-simple Lie algebras, one makes the following

Definition 2.1. A representation V of the Yangian Y is said to be *highest weight* if there is a vector $\Omega \in V$ such that $V = Y\Omega$ and

$$x_k^+ \Omega = 0, \qquad h_k \Omega = d_k \Omega, \qquad k = 0, 1, \dots$$

for some sequence of complex numbers $\mathbf{d} = (d_0, d_1, \dots)$. In this case, Ω is called a highest weight vector of V and $\mathbf{d}$ its highest weight.

Remark. It follows immediately from Definition 1.1 that the assignment $x \mapsto x$ for $x \in \mathfrak{sl}_2$ extends to a homomorphism of algebras $\iota\colon U(\mathfrak{sl}_2) \to Y$. By Proposition 2.5 below, ι is injective. Thus, any representation of Y can be restricted to give a representation of $\mathfrak{sl}_2$. In particular, we can speak of weights relative to $\mathfrak{sl}_2$ as well as relative to Y. It will always be clear from the context which type of weight is intended.

As in the case of semi-simple Lie algebras, there is a universal highest weight representation of Y of any given highest weight:

Definition 2.2. Let $\mathbf{d} = (d_0, d_1, \dots)$ be any sequence of complex numbers. The *Verma representation* $M(\mathbf{d})$ is the quotient of Y by the left ideal generated by $\{x_k^+, h_k - d_k \cdot 1\}_{k \in \mathbf{Z}_+}$.

PROPOSITION 2.3. *The Verma representation* $M(\mathbf{d})$ *is a highest weight representation with highest weight* $\mathbf{d}$, *and every such representation is* *isomorphic to a quotient of* $M(\mathbf{d})$. *Moreover,* $M(\mathbf{d})$ *has a unique irreducible quotient* $V(\mathbf{d})$.

Proof. Only the last statement requires proof. We consider $M(\mathbf{d})$ as a representation of $\mathfrak{sl}_2$. By Proposition 1.11, the d_0-weight space $\{v \in M(\mathbf{d}) : h_0.v = d_0 v\}$ is one-dimensional, and spanned by the highest weight vector $1 \in M(\mathbf{d})$. Thus, if M_1 and M_2 are two proper subrepresentations of $M(\mathbf{d})$, then $M_1 + M_2$ is also proper. It follows that $M(\mathbf{d})$ has a unique maximal proper subrepresentation.

The question of which highest weight representations are finite-dimensional was answered by Drinfel'd in [5, Theorem 2]. His result may be stated as follows.

THEOREM 2.4. *(a) Every irreducible finite-dimensional representation of* Y *is highest weight.*

(b) The irreducible highest weight representation $V(\mathbf{d})$ *of* Y *is finite-dimensional if and only if there exists a monic polynomial* $P \in \mathbf{C}[u]$ *such that*

$$\frac{P(u+1)}{P(u)} = 1 + \sum_{k=0}^{\infty} d_k u^{-k-1},$$

in the sense that the right-hand side is the Laurent expansion of the left-hand side about $u = \infty$.

To construct examples of highest weight representations of Y, we need the following result, which is an immediate consequence of the defining relations (1.1).

PROPOSITION 2.5. *(a) The assignment* $x \mapsto x,\ J(x) \mapsto 0$ *extends to a homomorphism of algebras* $\varepsilon_0 : Y \to U(\mathfrak{sl}_2)$.

(b) For any $a \in \mathbf{C}$, *the assignment* $x \mapsto x,\ J(x) \mapsto J(x) + ax$ *extends to an automorphism* τ_a *of* Y.

By part (a), if V is a representation of $\mathfrak{sl}_2$, one can pull it back by ε_0 to give a representation V of Y. Pulling back this representation by τ_a then gives a one-parameter family of representations $V(a)$ of Y. Note that $V(a)$ is an irreducible representation of Y because ε_0 is surjective.

Let W_m be the $(m+1)$-dimensional irreducible representation of $\mathfrak{sl}_2$, $m \in \mathbf{Z}_+$. Then, $W_m(a)$ has a basis $\{e_0, \ldots, e_m\}$ on which the action of Y is given by:

$$x^+.e_i = (i+1)e_{i+1}, \qquad x^-.e_i = (m-i+1)e_{i-1}, \qquad h.e_i = (2i-m)e_i,$$

the action of $J(h)$ (resp. $J(x^{\pm})$) being a times that of h (resp. $x^{\pm}$). To make contact with the theory of highest weight representations, we need:

PROPOSITION 2.6. *The action of the generators* $h_k, x_k^{\pm}$ *on* $W_m(a)$ *is given by:*

(1) $x_k^+ . e_i = \left(a - \frac{1}{2}m + i + \frac{1}{2}\right)^k (i+1)e_{i+1}$;

(2) $x_k^- . e_i = \left(a - \frac{1}{2}m + i - \frac{1}{2}\right)^k (m-i+1)e_{i-1}$;

(3)
$$h_k . e_i = \left\{\left(a - \frac{1}{2}m + i - \frac{1}{2}\right)^k i(m-i+1) - \left(a - \frac{1}{2}m + i + \frac{1}{2}\right)^k (i+1)(m-i)\right\} e_i .$$

Proof. It is straightforward to check, using the relations (1)-(3) in Theorem 1.2, that these formulas do define a representation of Y. It therefore suffices to check that they also give the correct action of the generators h, $J(h)$, $x^{\pm}$, $J(x^{\pm})$. This is another straightforward computation, using the isomorphism ϕ in (1.2).

COROLLARY 2.7. *(a)* $W_m(a)$ *is a highest weight representation with highest weight* $\mathbf{d} = (d_0, d_1, \ldots)$ *given by*

$$d_k = m\left(a + \frac{1}{2}m - \frac{1}{2}\right)^k .$$

(b) The monic polynomial P *associated to* $W_m(a)$ *is given by*

$$P(u) = \left(u - a + \frac{1}{2}m - \frac{1}{2}\right)\left(u - a + \frac{1}{2}m - \frac{3}{2}\right) \ldots \left(u - a - \frac{1}{2}m + \frac{1}{2}\right) .$$

Proof. (a) It is clear that e_m is a highest weight vector for $W_m(a)$ relative to Y. The eigenvalues of the h_k on e_m are as stated.

(b) By Theorem 2.4(b), the polynomial P is determined by

$$\frac{P(u+1)}{P(u)} = 1 + \sum_{k=0}^{\infty} m\left(a + \frac{1}{2}m - \frac{1}{2}\right)^k u^{-k-1}$$

$$= \frac{\left(u - a + \frac{1}{2}m + \frac{1}{2}\right)}{\left(u - a - \frac{1}{2}m + \frac{1}{2}\right)} .$$

The stated P clearly satisfies this equation.

In section 4 we shall need to consider the duals of the evaluation representations $W_m(a)$. If V is any finite-dimensional representation of Y, its dual V^* is naturally a representation of Y^{op}, the vector space Y with the opposite multiplication:

$$x.y\,(in\ Y^{op}) = y.x\,(in\ Y) .$$

Moreover, Y^{op} is a Hopf algebra with the same co-multiplication as Y.

PROPOSITION 2.8. *There is an isomorphism of Hopf algebras* $\theta: Y \to Y^{op}$ *such that*

$$\theta(x) = -x, \qquad \theta(J(x)) = J(x)$$

for all $x \in \mathfrak{sl}_2$.

Proof. It is sufficient to prove that the assignment $x \mapsto -x$, $J(x) \mapsto J(x)$ extends to a homomorphism of Hopf algebras $Y \to Y^{op}$. The relations in Y^{op} are obtained by inserting a minus sign on the right-hand side of relations (1) and (3) in (1.1). The result is now clear.

Remark. The anti-homomorphism $\theta: Y \to Y$ is closely related to the antipode S of Y, which is given by

$$S(x) = -x, \qquad S(J(x)) = -J(x) + \frac{1}{4}cx ,$$

where c is the eigenvalue of the Casimir operator in the adjoint representation of $\mathfrak{sl}_2$ (which depends of course on the choice of inner product (,) on $\mathfrak{sl}_2$).

Thus, if V is a finite-dimensional representation of Y, then V^* is a representation of Y with action

$$(y.f)(v) = f(\theta(y).v) ,$$

for $y \in Y$, $v \in V$ and $f \in V^*$. Moreover, the fact that θ preserves the co-multiplication implies that $(V_1 \otimes V_2)^* \cong V_1^* \otimes V_2^*$ for any two representations V_1, V_2 of Y.

COROLLARY 2.9. *As representations of* Y, *we have*

$$W_m(a)^* \cong W_m(-a)\ .$$

Proof. On $W_m(a)$, $J(x)$ acts as ax. Therefore, on $W_m(a)^*$, $J(x)$ acts as $-ax$.

The following is a related result.

PROPOSITION 2.10. *Every evaluation representation* $W_m(a)$ *has a non-degenerate invariant symmetric bilinear form.*

This means that there is a non-degenerate symmetric bilinear form $<\ ,\ >$ on $W_m(a)$ such that

$$<y.v_1, v_2> \;=\; <v_1, \omega(y).v_2> \tag{2.11}$$

for all $y \in Y$, $v_1, v_2 \in W_m(a)$.

Proof. It is well-known that the representation W_m of $\mathfrak{sl}_2$ carries a form $<\ ,\ >$ which satisfies (2.11) for all $y \in \mathfrak{sl}_2$. Moreover, the form is unique up to a scalar multiple because W_m is irreducible. To prove (2.11) in general, it suffices to check the case $y = x_k^+$, since the case $y = x_k^-$ then follows because $<\ ,\ >$ is symmetric, and $\omega(x_k^+) = x_k^-$. Since vectors of different weights are orthogonal, it is therefore enough to prove.

$$<x_k^+ . e_i, e_{i+k}> \;=\; <e_i, x_k^- . e_{i+k}> \tag{2.12}$$

(with the understanding that $e_i = 0$ unless $0 \leqslant i \leqslant n$). This follows easily from Proposition 2.6 and the invariance of $<\ ,\ >$ under $\mathfrak{sl}_2$.

3. A COMBINATORIAL INTERLUDE

The form of the polynomial P associated to the representation $W_m(a)$ in Corollary 2.7(b) suggests the following definition.

Definition 3.1. A non-empty finite set of complex numbers is said to be a *string* if it is of the form $\{a, a+1, \ldots, a+n\}$ for some $a \in \mathbf{C}$ and some $n \in \mathbf{N}$. The centre of the string is $a + \dfrac{n}{2}$ and its length is $n+1$.

We shall also need:

Definition 3.2. Two strings S_1 and S_2 are said to be *non-interacting* if either