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by Roland BACHER

0. INTRODUCTION

We will consider the standard lattice $\mathbf{Z}^n$ of the real vector space $\mathbf{R}^n$ with $n \geqslant 2$. An *integral simplex* is a non-degenerate simplex of $\mathbf{R}^n$ with all vertices in $\mathbf{Z}^n$. In this note, all simplices will be integral.

We will denote by $\mathrm{Aff}(\mathbf{Z}^n)$ the group of affine bijections of $\mathbf{R}^n$ which preserve $\mathbf{Z}^n$; it is the usual semi-direct product $\mathbf{Z}^n \rtimes GL_n(\mathbf{Z})$. The affine groupe $\mathrm{Aff}(\mathbf{Z}^n)$ acts naturally on the set of integral simplices in $\mathbf{Z}^n$.

For each integral simplex S we define

$$\mathrm{Stab}(S) = \{g \in \mathrm{Aff}(\mathbf{Z}^n) \mid g(S) = S\}$$

which is of course a subgroup of the group σ_{n+1} (group of permutations of $n+1$ objects), since there exists an injection in the group of permutations of the vertices of S.

Definition 0.1. A simplex S is called *affinely regular* if $\mathrm{Stab}(S)$ is equal to the whole group σ_{n+1}.

The definition of an affinely regular simplex is independent of the metric. For a discussion of integral simplices which are metrically regular one can consult [1] or [2] of the bibliography.

Two simplices S and S' are *equivalent* if there exists $g \in \mathrm{Aff}(\mathbf{Z}^n)$ such that $g(S) = S'$. The scope of this note is to find all equivalence classes of affinely regular simplices.

Let S be a simplex. Let us denote by λS the image of the simplex S multiplied by some non-zero integer λ.

PROPOSITION 0.2. *The groups* $\mathrm{Stab}(S)$ *and* $\mathrm{Stab}(\lambda S)$ *are isomorphic for any integer* $\lambda \neq 0$.

Proof. Denote by $\delta(\lambda)$ the linear automorphism $x \mapsto \lambda x$ of $\mathbf{R}^n$. Let ϕ_λ denote the endomorphism $g \mapsto \delta(\lambda) g \delta(\lambda^{-1})$ of $\mathrm{Aff}(\mathbf{Z}^n)$; observe that ϕ_λ is one-to-one, but is not onto if $|\lambda| \geqslant 2$. Indeed, an affine bijection $g \in \mathrm{Aff}(\mathbf{Z}^n)$ is in the image of ϕ_λ if and only if g preserves the sublattice $\lambda\mathbf{Z}^n$ of $\mathbf{Z}^n$.

If $g \in \mathrm{Stab}(S)$, then $\phi_\lambda(g) \in \mathrm{Stab}(\lambda S)$. Consequently ϕ_λ restricts to an injective homomorphism $\psi_\lambda \colon \mathrm{Stab}(S) \to \mathrm{Stab}(\lambda S)$. Let now $h \in \mathrm{Stab}(\lambda S)$. We can write $h = at$, where a is in $GL_n(\mathbf{Z})$ and where t is a translation. As a^{-1} preserves $\lambda\mathbf{Z}^n$ (as any element of $GL_n(\mathbf{Z})$ does), and as h preserves λS one has

$$t(\lambda S) = a^{-1}h(\lambda S) = a^{-1}(\lambda S) \subset a^{-1}\lambda\mathbf{Z}^n$$

so that t preserves $\lambda\mathbf{Z}^n$. Hence $h = at$ preserves $\lambda\mathbf{Z}^n$, so that h is in the image of ϕ_λ. It follows that ψ_λ is an isomorphism onto. □

Caution: We have in fact proved that $\mathrm{Stab}(S)$ and $\mathrm{Stab}(\lambda S)$ are conjugate in $\mathrm{Aff}(\mathbf{Q}^n)$ but they are in general not conjugate in $\mathrm{Aff}(\mathbf{Z}^n)$. This can be seen for instance by the fact that $\mathrm{Stab}(S)$ fixes the barycenter P of S and $\mathrm{Stab}(\lambda S)$ fixes λP. But P and λP are not necessarily in the same orbit of $\mathrm{Aff}(\mathbf{Z}^n)$.

So λS is affinely regular if and only if S is affinely regular. Hence we will be interested in minimal simplices.

Definition 0.3. An integral simplex S is *minimal* if, for every integral simplex T and for every integer $\lambda \geqslant 1$ such that S is equivalent to λT, we have $\lambda = 1$.

PROPOSITION 0.4. *Let S be an integral simplex of $\mathbf{Z}^n$. The following assertions are equivalent:*

i) *S is minimal.*

ii) *For every integer $\lambda \geqslant 2$ there exists no class of $\mathbf{Z}^n$ modulo $\lambda\mathbf{Z}^n$ which contains all the vertices of S modulo $\lambda\mathbf{Z}^n$.*

Proof. Not (ii) $\Rightarrow$ not (i). Let S be a simplex with all vertices in the same class of $\mathbf{Z}^n$ modulo $\lambda\mathbf{Z}^n$. Let v_0 be one of the vertices. The translate of S by $-v_0$ is then a simplex with the coordinates of all vertices divisible by some $\lambda \geqslant 2$. This implies that S is not minimal.

Not (i) $\Rightarrow$ not (ii). Let S be a non-minimal integral simplex. Hence there exists an integral simplex T, an integer $\lambda \geqslant 2$, an element $g \in GL_n(\mathbf{Z})$ and a vector $v \in \mathbf{Z}^n$ such that $S = g(\lambda T) + v$. But then all the vertices of S are in the class of v in $\mathbf{Z}^n$ modulo $\lambda\mathbf{Z}^n$. □

The main subject of this note is to show the following theorem:

THEOREM 0.5. *For* $n \geqslant 2$, *one has a bijection between the equivalence classes of minimal affinely regular integral simplices and the set of positive divisors of* $n+1$ *(including* 1 *and* $n+1$*). The bijection associates to the divisor* k *of* $n+1$ *the class of the simplex whose vertices are given by the columns of the following* $n \times n$ *matrix*

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & k-1 \\ 0 & 1 & 0 & \dots & 0 & k-1 \\ . & . & . & \dots & . & k-1 \\ 0 & 0 & 0 & \dots & 1 & k-1 \\ 0 & 0 & 0 & \dots & 0 & k \end{pmatrix}$$

and by the origin of $\mathbf{Z}^n$.

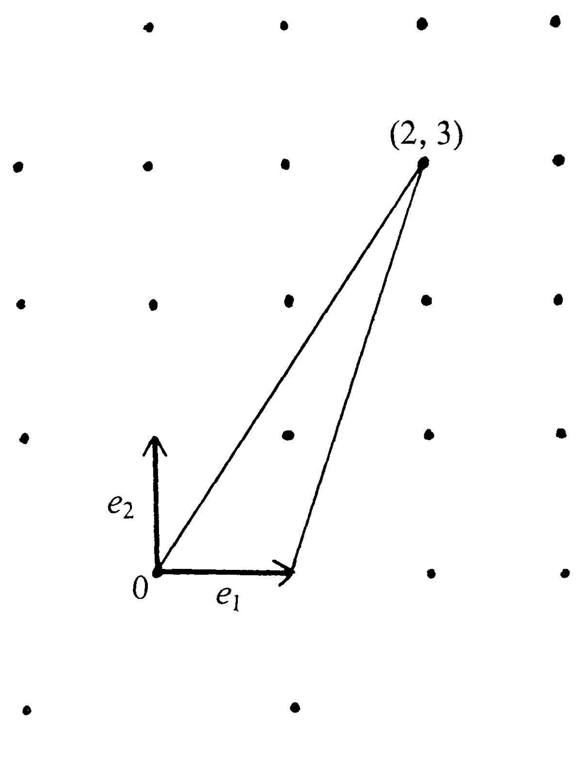
Proposition 0.4 implies that all representants in the theorem are minimal. Moreover, representants associated to distinct divisors k, k' of $n+1$ are non-equivalent since they are respectively of volumes $k/n!$ and $k'/n!$.

The plan of the proof is as follows. We will introduce a family of particular simplices: those which have small faces. Then we dress the list of all small-faced affinely regular simplices (this gives us in fact the list of the theorem). Last, we prove that an affinely regular minimal simplex is necessarily small-faced.

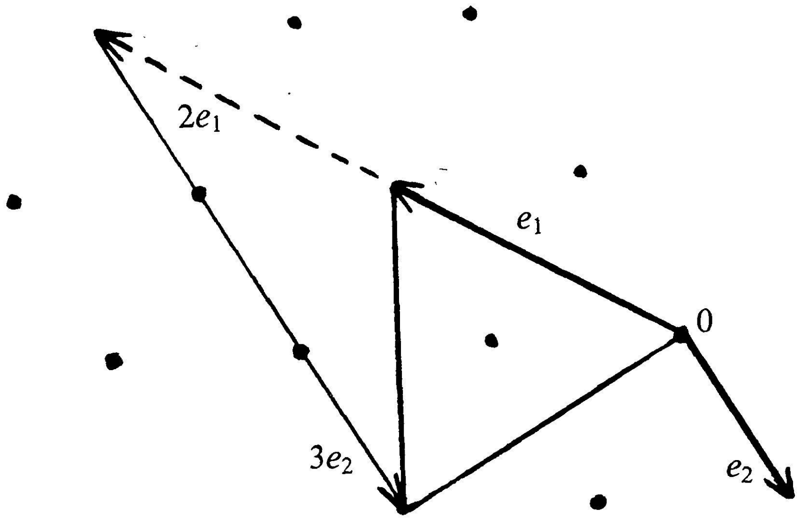
Let us start with some examples:

Example 0.6. Case where $n = 2, k = 3$.

In the standard lattice:



In the hexagonal lattice:



Example 0.7. Case where $n = 3, k = 2$.

Let $C = [0,1]^3$ be the standard cube of $\mathbf{R}^3$. Let Δ be the tetrahedron defined by the vertices of the cube of which the sum of the coordinates is even. It is easy to see that Δ is affinely regular and that the linear transformation defined by

$$e_1 \mapsto -e_3\ , \quad e_2 \mapsto e_1 + e_3\ , \quad e_3 \mapsto e_2 + e_3$$

(where (e_1, e_2, e_3) is the standard basis of $\mathbf{R}^3$) sends Δ to the representant given in Theorem 0.5.

1. Simplices with small faces

Definition 1.1. An integral simplex S is said to have *small faces* if, for each hyperplan H containing a $(n-1)$-face of S, the vertices of S contained in H constitute an affine $\mathbf{Z}$-basis of $\mathbf{Z}^n \cap H$.

A *numerotation* of an integral simplex S is an enumeration

$$\nu = (v_0, v_1, .., v_n)$$

of the vertices of S. We will denote by S_ν the simplex S with numerotation ν. The group $\mathrm{Aff}(\mathbf{Z}^n)$ acts naturally on the set of numerated simplices and we will say that S_{ν} (with $\nu = (v_0, .., v_n)$) is equivalent to $S'_{\nu'}$ (with $\nu' = (v'_0, .., v'_n)$) if there exists $g \in \mathrm{Aff}(\mathbf{Z}^n)$ such that $g(v_i) = v'_i$ for all i.

The group σ_{n+1} acts on the set of numerated simplices: If $s \in \sigma_{n+1}$ is a permutation of $\{0, .., n\}$ and if S_{ν} is an integral simplex numerated by $\nu = (v_0, .., v_i, .., v_n)$, we define

$$s \cdot S_{\nu} = S_{s \cdot \nu}$$

where

$$s \cdot \nu = (v_{s^{-1}(0)}, .., v_{s^{-1}(i)}, .., v_{s^{-1}(n)}) .$$

This action of σ_{n+1} commutes with that of $\mathrm{Aff}(\mathbf{Z}^n)$, hence σ_{n+1} acts also on the equivalence classes of numerated simplices modulo $\mathrm{Aff}(\mathbf{Z}^n)$.

An integral simplex S is affinely regular if and only if the stabilizer $\mathrm{Stab}(S)$ operates transitively on the numerotations of S.

Let us recall an elementary and well-known lemma:

LEMMA 1.2. *Let* $v_1, .., v_{n-1}$ *be linearly independent vectors of* $\mathbf{Z}^n$. *Let* H *be the hyperplane of* $\mathbf{R}^n$ *generated by the* v_i's, *and suppose that* $v_1, .., v_{n-1}$ *form a basis of the sublattice* $H \cap \mathbf{Z}^n$.

Then we can complete $v_1, .., v_{n-1}$ *to a basis of* $\mathbf{Z}^n$.

Proof. See for instance the Corollary in Bourbaki, *Algèbre,* chap. VII, §4, No. 3.

From this point until the end of section 2, k will be some fixed natural integer. Let now S_{ν} be a numerated simplex with small faces, of volume $k/n!$. Let $\{e_1, .., e_n\}$ be the canonical basis of $\mathbf{Z}^n \subset \mathbf{R}^n$. The lemma implies that there exists $g \in \mathrm{Aff}(\mathbf{Z}^n)$ such that gS_{ν} has vertices

$$v_0 = 0, v_1 = e_1, .., v_{n-1} = e_{n-1}, v_n = ke_n + \sum_{i=1}^{n-1} a_i e_i$$

where the a_i's are integers. An easy calculation shows that the a_i's are well defined (mod k).

Let us associate to S_{ν} the element $(\alpha_1, .., \alpha_{n-1})$, where α_i is the class of a_i (mod k). This gives us a map $\rho_k \colon \Sigma_k \to (\mathbf{Z}/k\mathbf{Z})^{n-1}$, where Σ_k is the set of numerated simplices with small faces of volume $k/n!$. For $S_{\nu} \in \Sigma_k$, the element $\rho_k(S_{\nu})$ depends only of the equivalence class of S_{ν} modulo $\mathrm{Aff}(\mathbf{Z}^n)$ and this allows us do define an action of σ_{n+1} on $\rho_k(\Sigma_k)$.

2. Action of σ_{n+1} on $\rho_k(\Sigma_k)$

In this section, we show how the n usual generators $(0,1), \ldots, (n-1,n)$ of the group σ_{n+1} act on the subset $\rho_k(\Sigma_k)$ of $(\mathbf{Z}/k\mathbf{Z})^{n-1}$.

Lemma 2.1. *Let* $(\alpha_1, .., \alpha_{n-1}) \in \rho_k(\Sigma_k)$. *Then*

$$(0,1) \cdot (\alpha_1, \alpha_2, .., \alpha_i, .., \alpha_{n-1}) = ((1 - \alpha_1 - \alpha_2 - .. - \alpha_{n-1}), \alpha_2, .., \alpha_i, .., \alpha_{n-1})$$

$$(i, i+1) \cdot (\alpha_1, .., \alpha_i, \alpha_{i+1}, .., \alpha_{n-1}) = (\alpha_1, .., \alpha_{i+1}, \alpha_i, .. \alpha_{n-1}), \quad 1 \leqslant i \leqslant n-2$$

$$(n-1, n) \cdot (\alpha_1, .., \alpha_i, .., \alpha_{n-2}, \alpha_{n-1}) = (-\alpha_1 \alpha_{n-1}^{-1}, .., -\alpha_i \alpha_{n-1}^{-1}, .., -\alpha_{n-2} \alpha_{n-1}^{-1}, \alpha_{n-1}^{-1}$$

In particular, if $(\alpha_1, .., \alpha_{n-1}) \in \rho_k(\Sigma_k)$, *then* α_{n-1} *is invertible* (mod k).

Proof. Let us show the first equality. Let $(\alpha_1, .., \alpha_{n-1}) \in \rho_k(\Sigma_k)$. Consider the numerated simplex S_v with vertices

$$v_0 = 0, v_i = e_i, 1 \leqslant i \leqslant n-1, v_n = ke_n + \sum_{i=1}^{n-1} a_i e_i \tag{2.2}$$

(a_i representant of the class α_i).

Let us identify $\mathbf{R}^n$ with the points of the hyperplane H of $\mathbf{R}^{n+1}$ defined by $x_{n+1} = 1$. By identifying the elements of $\mathbf{R}^n$ with vector columns we see that

$$0 \mapsto \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{pmatrix}, \quad e_i \mapsto \begin{pmatrix} 0 \\ \cdot \\ 1 \\ \cdot \\ 0 \\ 1 \end{pmatrix}$$

where 0 is the origin in $\mathbf{R}^n$ and $\{e_1, \ldots, e_n\}$ is the usual basis of $\mathbf{R}^n$ and the 1 in the second matrix is where you think it should be, namely at the i-th row. There exists a natural injection of the group $\mathrm{Aff}(\mathbf{Z}^n)$ in the subgroup of $GL_{n+1}(\mathbf{Z})$ which preserves H. It is easy to check that the following matrix $M \in GL_{n+1}(\mathbf{Z})$ exchanges v_0 and v_1, preserves v_i for $2 \leqslant i < n$ and sends v_n to the element $ke_n + (1 - \sum_{i=1}^{n-1} a_i) e_1 + \sum_{i=2}^{n-1} a_i e_i$:

$$M = \begin{pmatrix} -1 & -1 & -1 & .. & -1 & 0 & 1 \\ 0 & 1 & 0 & .. & 0 & 0 & 0 \\ 0 & 0 & 1 & .. & 0 & 0 & 0 \\ .. & . & . & .. & . & . & . \\ 0 & 0 & 0 & .. & 1 & 0 & 0 \\ 0 & 0 & 0 & .. & 0 & 1 & 0 \\ 0 & 0 & 0 & .. & 0 & 0 & 1 \end{pmatrix}.$$

The calculations for the transposition $(i, i+1)$ are immediate if $1 \leqslant i \leqslant n-2$.

Finally, let us consider the last equality: We take again the simplex S_ν with vertices as in (2.2). Since S_ν is small-faced, there exists a simplex S'_μ, $(n-1)$ integers $b_1, .., b_{n-1}$ and an element $g \in \mathrm{Aff}(\mathbf{Z}^n)$ such that the vertices of S'_μ are

$$v'_0 = 0, v'_i = e_i, 1 \leqslant i \leqslant n-1, v'_n = ke_n + \sum_{i=1}^{n-1} b_i e_i$$

and such that $g(v_i) = v'_i$ for $0 \leqslant i \leqslant n-2$, $g(v_{n-1}) = v'_n$ and $g(v_n) = v'_{n-1}$. Since $v_0 = v'_0 = 0$ we have $g(0) = 0$ and g is in fact in $GL_n(\mathbf{Z})$. The matrix of g with respect to the standard basis is

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & b_1 \\ 0 & 1 & 0 & \dots & 0 & 0 & b_2 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 & 0 & b_{n-2} \\ 0 & 0 & 0 & \dots & 0 & 1 & b_{n-1} \\ 0 & 0 & 0 & \dots & 0 & 0 & k \end{pmatrix} \begin{pmatrix} 1 & 0 & \dots & 0 & a_1 & 0 \\ 0 & 1 & \dots & 0 & a_2 & 0 \\ \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 1 & a_{n-2} & 0 \\ 0 & 0 & \dots & 0 & a_{n-1} & 1 \\ 0 & 0 & \dots & 0 & k & 0 \end{pmatrix}^{-1}$$

this gives

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & b_1 \\ 0 & 1 & 0 & \dots & 0 & 0 & b_2 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 & 0 & b_{n-2} \\ 0 & 0 & 0 & \dots & 0 & 1 & b_{n-1} \\ 0 & 0 & 0 & \dots & 0 & 0 & k \end{pmatrix} \frac{1}{k} \begin{pmatrix} k & 0 & 0 & \dots & 0 & 0 & -a_1 \\ 0 & k & 0 & \dots & 0 & 0 & -a_2 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & k & 0 & -a_{n-2} \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & k & -a_{n-1} \end{pmatrix}.$$

But since $g \in GL_n(\mathbf{Z})$ this implies that

$$-a_i - b_i a_{n-1} \equiv 0 \pmod k \qquad 1 \leqslant i \leqslant n-2\,.$$

Let us now suppose that a_{n-1} is not invertible (mod k). Then there exists some prime p dividing both k and a_{n-1}. But then the prime p divides a_i too for every i. So p divides all coefficients of the vector $v_n - v_0$ which is an edge of S_ν. But then S_ν is not small-faced which contradicts the fact that $(\alpha_1, .., \alpha_{n-1}) \in \rho_k(\Sigma_k)$. So we have proved that a_{n-1} is invertible (mod k). And the b_i's satisfy

$$b_i \equiv -a_i a_{n-1}^{-1} \pmod k \qquad 1 \leqslant i \leqslant n-2 \quad \text{and} \quad b_{n-1} \equiv a_{n-1}^{-1} \pmod k\,.$$

This proves the last equality. $\square$

3. LIST OF SMALL-FACED AFFINELY REGULAR SIMPLICES

An integral simplex S with small faces is affinely regular if and only if the numerated simplices S_ν and $S_{\nu'}$ are equivalent for each pair ν, ν' of enumerations of S. In other terms, an integral simplex S with small faces of volume $k/n!$ is affinely regular if and only if $\rho_k(S_\nu) = \rho_k(S_{\nu'})$ for all enumerations ν and ν', hence if and only if the element $\rho_k(S_\nu)$ is a fixed point under the action of σ_{n+1} on $\rho_k(\Sigma_k)$.

It is sufficient for $\rho_k(S_\nu)$ to be fixed under the action of a set of generators in order to be a fixpoint of σ_{n+1} acting on $\rho_k(\Sigma_k)$. Let us suppose that $(\alpha_1, .., \alpha_{n-1})$ is a fixpoint of $\rho_k(\Sigma_k)$. Then, for all $i \in \{1, ..., n-2\}$:

$$\begin{aligned}(i, i+1) \cdot (\alpha_1, .., \alpha_i, \alpha_{i+1}, .., \alpha_{n-1}) &= (\alpha_1, .., \alpha_{i+1}, \alpha_i, .. \alpha_{n-1}) \\ &= (\alpha_1, .. \alpha_i, \alpha_{i+1}, .., \alpha_{n-1})\end{aligned}$$

implies $\alpha_i \equiv \alpha \pmod{k}$ for some $\alpha \in \mathbf{Z}/k\mathbf{Z}$.

Furthermore

$$(n-1, n) \cdot (\alpha, .., \alpha, \alpha) = (-\alpha\alpha^{-1}, .., -\alpha\alpha^{-1}, \alpha^{-1}) = (\alpha, .., \alpha)$$

gives $\alpha \equiv -\alpha\alpha^{-1} \equiv -1 \pmod{k}$.

Finally

$$(0, 1) \cdot (-1, -1, .., -1) = (1-(n-1)(-1), -1, .., -1) = (-1, .., -1)$$

implies $-1 \equiv 1-(n-1)(-1) \pmod{k}$ namely $0 \equiv n+1 \pmod{k}$ namely $k \mid (n+1)$.

This shows that the simplices listed in the theorem are exactly all the affinely regular simplices with small faces.

We have yet to show that any affinely regular minimal simplex is small-faced. This will be the aim of the next paragraph.

4. ANY AFFINELY REGULAR MINIMAL SIMPLEX HAS SMALL FACES

Lemma 1.2 implies the following corollary:

COROLLARY 4.1. *Every integral simplex of* $\mathbf{Z}^n$ *with numerated vertices is equivalent to an integral simplex with vertex* v_0 *at* 0 *and vertex* v_i *at the i-th vector-column of an upper triangular matrix* $(i > 0)$.

Regularity and Proposition 0.4 imply almost immediately the following:

Remark 4.2. Let S be an affinely regular simplex. Then S is minimal if and only if the interior of each edge of S is without integral points.

Let us start with the proof that each affinely regular minimal simplex is small-faced.

Consider an affinely regular minimal simplex S of $\mathbf{Z}^2$. Corollary 4.1 implies that S is equivalent to a simplex S' with vertices at 0 and at the vector-columns of a matrix of the type $\begin{pmatrix} l & a \\ 0 & k \end{pmatrix}$. Remark 4.2 implies that the integer l is equal to ± 1.

By exchanging S' with an equivalent simplex if necessary, we can suppose that S' has its vertices at 0 and at the vector-columns of a matrix of type $\begin{pmatrix} 1 & a \\ 0 & k \end{pmatrix}$ with k a positive integer. The affine regularity now implies that S' (and hence S) is small-faced.

Hence the theorem holds for $n = 2$.

Induction: $(n-1) \Rightarrow (n)$.

Let S_ν be a numerated affinely regular minimal simplex of $\mathbf{Z}^n$ with underlying simplex S. Corollary 4.1 implies that, after some suitable choice of an equivalent simplex, we can suppose that $v_0 = 0$ and v_i is the i-th vector-column of an upper triangular $n \times n$ matrix T.

The $(n-1)$-face containing $v_0, v_1, .., v_{n-1}$ is an affinely regular simplex of $\mathbf{Z}^{n-1}$ and Remark 4.2 shows that it is minimal. So Lemma 1.2 and the induction hypothesis imply that, possibly after a suitable change of S_ν, the matrix T is of the form

$$\begin{pmatrix} 1 & 0 & 0 & .. & 0 & l-1 & a_1 \\ 0 & 1 & 0 & .. & 0 & l-1 & a_2 \\ . & . & . & .. & . & . & . \\ 0 & 0 & 0 & .. & 1 & l-1 & a_{n-2} \\ 0 & 0 & 0 & .. & 0 & l & a_{n-1} \\ 0 & 0 & 0 & .. & 0 & 0 & k \end{pmatrix}$$

where l and k are positive integers, and where l divides n by induction hypothesis.

Set $\mu = n/l \in \mathbf{N}$.

The barycenter of $\mu v_0, \mu v_1, .., \mu v_{n-1}$ is $e_1 + e_2 + .. + e_{n-1}$.

Since S_ν is affinely regular, there exists for all $i \in \{0, 1, .., n\}$ an element $g_i \in \mathrm{Aff}(\mathbf{Z}^n)$ which sends

$$\mu v_0, .., \mu v_{n-1}, \widehat{\mu v_n} \quad \text{to} \quad \mu v_0, .., \widehat{\mu v_i}, .., \mu v_n \,;$$

the barycenter of $\mu v_0, .., \widehat{\mu v_i}, .., \mu v_n$, which is $g_i(e_1 + e_2 + .. + e_{n-1})$, is consequently also in $\mathbf{Z}^n$.

So the barycenters of all faces of μS_v are in $\mathbf{Z}^n$ and they are the vertices of an integral simplex S'.

Calculating the first coordinate of the barycenter of $\widehat{\mu v_0}, \mu v_1, .., \mu v_n$ we see that n divides $\mu + \mu(l-1) + \mu a_1$.

Calculating the first coordinate of the barycenter of $\mu v_0, \widehat{\mu v_1}, \mu v_2, .., \mu v_n$, we see that n divides $\mu(l-1) + \mu a_1$.

So the integer n divides μ too but this implies that $\mu = n$ and hence $l = 1$. This and the affine regularity imply that S is small-faced. $\square$

The notions of affine regularity and of integrality may both be generalized to other polytopes, such as hypercubes, cross-polytopes, hexagones in dimension 2 or exceptional polytopes in dimension 4. We plan to consider these in a further paper.

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