The error term in (2) is best possible, and for the range $1/2 < \sigma < 3/4$ the above asymptotic formula has been considerably refined by K. Matsumoto [5] (see also [4]).

It seems that the case $\sigma = 1$ in (1) has not received much attention in the literature. The aim of this paper is to discuss this problem. We shall prove the following

THEOREM. *Let for* $T > 1$ *the function* $R(T)$ *be defined by*

$$\int_1^T |\zeta(1+it)|^2 dt = \zeta(2)T - \pi \log T + R(T) . \tag{3}$$

Then

$$R(T) = O(\log^{2/3} T(\log\log T)^{1/3}) , \tag{4}$$

$$\int_1^T R(t)dt = O(T) , \tag{5}$$

and

$$\int_1^T R^2(t)dt = O(T(\log\log T)^4)) . \tag{6}$$

Remarks. In view of (4) it is seen that $R(T)$ represents the error term in the mean square formula for $\zeta(s)$ on the line $\sigma = 1$.

One often takes the lower limit of integration in (1) as zero. However, in our case ($\sigma = 1$) this cannot be done, since $\zeta(s)$ has a pole at $s = 1$. If in (3) we take some other positive number as the lower limit of integration, then obviously the value of $R(T)$ will be changed by a constant only.

The method used in the proof of our theorem may be used to evaluate mean values of certain other zeta-functions on the line $\sigma = 1$. These will be dealt with elsewhere.

The upper bound in (4) contains information about the order of $\zeta(1 + iT)$. Namely, from the general inequalities proved in [1] and [2] it may be deduced that

$$\zeta^k(1+iT) \ll \log^{2/3} T(\log\log T)^{1/3} \int_{T+\delta}^{T+\delta} |\zeta(1+it)|^k dt + 1$$

for any fixed integer $k \geqslant 1$ and $\delta = (\log\log T/\log T)^{2/3}$. Hence we obtain

$$\zeta(1+iT) \ll \log^{2/3} T(\log\log T)^{1/3} .$$

This bound is close to the classical bound of I. M. Vinogradov (see Ch. 6 of [3])

$$\zeta(1+iT) \ll \log^{2/3} T\,,$$

which for more than 30 years is the sharpest one, and follows from (19).

In view of (5) it seems plausible to conjecture that, as $T \to \infty$,

$$\int_1^T R^2(t)\,dt \sim AT$$

for some $A > 0$.

Acknowledgement. We are very much indebted to the referee for his remarks concerning the proof of (4).

Proof of the Theorem. We start from the simplest approximate functional equation for $\zeta(s)$ (see Th. 1.8 of [3]). Namely, for $0 < \sigma_0 \leqslant \sigma \leqslant 2$, $x \geqslant |t|/\pi$, $s = \sigma + it$, we have

$$\zeta(s) = \sum_{n \leqslant x} n^{-s} + \frac{x^{1-s}}{s-1} + O(x^{-\sigma})\,.$$

For $t \leqslant T$ this gives

$$\zeta(1+it) = \sum_{n \leqslant T} n^{-1-it} + \frac{T^{-it}}{it} + O\left(\frac{1}{T}\right). \tag{7}$$

Since $|z|^2 = z\bar{z}$ we obtain from (7)

$$\int_1^T |\zeta(1+it)|^2\,dt = \int_1^T \Big|\sum_{n \leqslant T} n^{-1-it}\Big|^2 dt - 2\operatorname{Re}\left\{\frac{1}{i}\int_1^T \sum_{n \leqslant T}\left(\frac{T}{n}\right)^{it}\frac{dt}{nt}\right\} \tag{8}$$

$$+\, O\left(\frac{1}{T}\int_1^T \Big|\sum_{n \leqslant T} n^{-1-it}\Big|\,dt\right) + O\left(\frac{\log T}{T}\right).$$

The main terms in the asymptotic formula (3) come from the first integral on the right-hand side of (8). To see this one can use the well-known Montgomery-Vaughan theorem for mean values of Dirichlet polynomials (see [6], also [7] and Ch. 5 of [3]), which says that

$$\int_0^T \Big|\sum_{n \leqslant N} a_n n^{-it}\Big|^2 dt = T\sum_{n \leqslant N} |a_n|^2 + O\Big(\sum_{n \leqslant N} n\,|a_n|^2\Big) \tag{9}$$

for arbitrary complex numbers a_n, $T > 0$ and $N \geqslant 1$. Therefore by the Cauchy-Schwarz inequality and (9) we have

$$\int_1^T \Big| \sum_{n \leqslant T} n^{-1-it} \Big| \, dt \leqslant T^{1/2} \left(\int_1^T \Big| \sum_{n \leqslant T} n^{-1-it} \Big|^2 dt \right)^{1/2} = O(T) . \tag{10}$$

We also have

$$\int_1^T \sum_{n \leqslant T} \left(\frac{T}{n}\right)^{it} \frac{dt}{nt} = O(\log\log T) . \tag{11}$$

To prove (11), let $2 \leqslant H \leqslant T/2$ be a parameter. Then

$$\int_1^T \sum_{n \leqslant T} \left(\frac{T}{n}\right)^{it} \frac{dt}{nt} = \sum_{n \leqslant T(1-1/H)} n^{-1} \int_1^T \left(\frac{T}{n}\right)^{it} \frac{dt}{t}$$

$$+ \sum_{T(1-1/H) < n \leqslant T} n^{-1} \int_1^T \left(\frac{T}{n}\right)^{it} \frac{dt}{t} = \sum_{n \leqslant T(1-1/H)} \frac{1}{n} \left\{ \frac{(T/n)^{it}}{it \log(T/n)} \Big|_1^T \right.$$

$$\left. + \int_1^T \frac{(T/n)^{it} dt}{it^2 \log(T/n)} \right\} + O\left(\sum_{T(1-1/H) < n \leqslant T} \frac{\log T}{n} \right)$$

$$= O\left(\sum_{n \leqslant T(1-1/H)} \frac{1}{n \log(T/n)} \right) + O\left(\frac{\log T}{H}\right)$$

$$\ll \int_1^{T(1-1/H)} \frac{dx}{x \log(T/x)} + \frac{\log T}{H} + 1 = \int_{(1-1/H)^{-1}}^T \frac{du}{u \log u} + \frac{\log T}{H} + 1$$

$$= \log\log T - \log\log\left(1 - \frac{1}{H}\right)^{-1} + \frac{\log T}{H} + 1 \ll \log\log T$$

for $H = \log T$. Thus from (8), (10) and (11) it follows that

$$\int_1^T |\zeta(1+iu)|^2 du = \int_1^T \Big| \sum_{n \leqslant T} n^{-1-iu} \Big|^2 du + O(\log\log T) . \tag{12}$$

Further we have

$$\int_1^T \Big| \sum_{n \leqslant T} n^{-1-iu} \Big|^2 du = \int_0^T \Big| \sum_{n \leqslant T} n^{-1-iu} \Big|^2 du - \int_0^1 \Big| \sum_{n \leqslant T} n^{-1-iu} \Big|^2 du \tag{13}$$

and observe that by the Euler-Maclaurin summation formula (or by (7)) we have, for $0 < u \leqslant 1$,

$$\sum_{n \leqslant T} n^{-1-iu} = \int_1^T x^{-1-iu}\,dx + \gamma + O(u) + O\left(\frac{1}{T}\right) = \frac{1 - T^{-iu}}{iu}$$

$$+ \gamma + O(u) + O\left(\frac{1}{T}\right),$$

where $\gamma = 0.5772157\ldots$ is Euler's constant. Therefore

$$\int_0^1 \Big| \sum_{n \leqslant T} n^{-1-iu} \Big|^2 du = \int_0^1 \frac{|1 - T^{-iu}|^2}{u^2}\,du - 2\gamma \operatorname{Re}\left\{ i \int_0^1 \frac{1 - T^{-iu}}{u}\,du \right\}$$

$$+ O(1) = 2\int_0^1 \frac{1 - \cos(u \log T)}{u^2}\,du + 2\gamma \int_0^1 \frac{\sin(u \log T)}{u}\,du + O(1) \tag{14}$$

$$= 4 \int_0^1 \frac{\sin^2((u \log T)/2)}{u^2}\,du + O(1) = 2 \log T \int_0^{(\log T)/2} \left(\frac{\sin v}{v}\right)^2 dv + O(1)$$

$$= \pi \log T + O(1),$$

since

$$\int_0^\infty \left(\frac{\sin v}{v}\right)^2 dv = \frac{\pi}{2}.$$

In (12) – (14) we replace T by t, and suppose that $T \leqslant t \leqslant 2T$. From (7) we have

$$\sum_{T < n \leqslant t} n^{-1-it} = O\left(\frac{1}{T}\right).$$

Hence from the definition of $R(T)$, given by (3), it follows that for $T \leqslant t \leqslant 2T$ we have

$$R(t) = \int_0^t \left(\Big| \sum_{n \leqslant T} n^{-1-iu} \Big|^2 - \zeta(2) \right) du + O(\log\log T). \tag{15}$$

This is the fundamental formula that will be used in the proof of (4) and (6). We start with the proof of (4), taking in (15) $t = T$ and writing

$$\sum_{n \leqslant T} n^{-1-iu} = \sum_{n \leqslant N} + \sum_{N < n \leqslant 2N} + \sum_{2N < n \leqslant T} = \Sigma_1 + \Sigma_2 + \Sigma_3,$$

say, where

$$N = \exp\left(C \log^{2/3} T (\log\log T)^{1/3}\right) \tag{16}$$

with a suitable constant $C > 0$. We have

$$\int_0^T \left| \sum\nolimits_2 + \sum\nolimits_3 \right|^2 du = \frac{1}{2} \int_{-T}^{T} \left| \sum_{N < n \leqslant T} n^{-1-iu} \right|^2 du \tag{17}$$

$$= T \sum_{N < n \leqslant T} n^{-2} + \sum_{N < m \neq n \leqslant T} \frac{(m/n)^{iT} - (m/n)^{-iT}}{2imn \log(m/n)} .$$

For the terms $n < m \leqslant 2n$ in the second sum on the right-hand side of (17) we may put $m = n + h$, producing a sum of the form

$$\sum_{1 \leqslant h \leqslant T} \sum_{\max(N,h) < n \leqslant T} \frac{(1 + h/n)^{iT} - (1 + h/n)^{-iT}}{2i(n+h)n \log(1 + h/n)} . \tag{18}$$

To estimate the inner sum over n we shall apply the Vinogradov-Korobov estimate for exponential (zeta) sums. In its original form this says that

$$\sum_{M < m \leqslant M_1 \leqslant 2M} m^{it} \ll M \exp\left(- \frac{C \log^3 M}{\log^2 t} \right) \quad \left(M_0 \leqslant M \leqslant \frac{1}{2} t, t \geqslant t_0 \right) , \tag{19}$$

and a proof (with $C = 10^{-5}$) may be found in Ch. 6 of [3]. However, it is easily seen that the method of proof of (19) yields also

$$\sum_{M < m \leqslant M_1 \leqslant 2M} \left(1 + \frac{h}{m} \right)^{it} \ll M \exp\left(- \frac{C \log^3 M}{\log^2 t} \right) \quad \left(M_0 \leqslant M_1 \leqslant \frac{1}{2} t, t \geqslant t_0 \right) \tag{20}$$

with some absolute $C > 0$, provided that $1 \leqslant h \ll m$. Therefore (20) gives

$$\sum_{N' < n \leqslant N''} \left(1 + \frac{h}{n} \right)^{iT} \ll N' \log^{-4} T \quad (N \leqslant N' < N'' \leqslant 2N' \leqslant T) ,$$

provided that the constant C in (16) is sufficiently large. It follows by partial summation that the sum in (18) is $\ll \log^{-2} T$. Moreover the terms with $m > 2n$ in (17) may be estimated directly by applying (19) to the sum over m, so that (17) becomes

$$\int_0^T \left| \sum\nolimits_2 + \sum\nolimits_3 \right|^2 du = T \sum_{N < n \leqslant T} n^{-2} + O(\log^{-2} T) . \tag{21}$$

Similarly we find that

$$\int_{-T}^{T} \sum_{n \leqslant N} n^{-1-iu} \sum_{2N < m \leqslant T} m^{-1+iu} du \ll \log^{-2} T \tag{22}$$

for N given by (16) and C sufficiently large. Now by (15) we have

$$(23) \quad R(T) = \frac{1}{2} \int_{-T}^{T} | \Sigma_1 + \Sigma_2 + \Sigma_3 |^2 du - \zeta(2)T + O(\log\log T),$$

and we shall use the elementary identity

$$| \Sigma_1 + \Sigma_2 + \Sigma_3 |^2 = | \Sigma_1 + \Sigma_2 |^2 - | \Sigma_2 |^2 + | \Sigma_2 + \Sigma_3 |^2 + 2\mathrm{Re}(\Sigma_1 \bar{\Sigma}_3).$$

By (9) the first two terms above contribute

$$T \sum_{n \leqslant N} n^{-2} + O(\log N)$$

to (23), while by (21) and (22) the contribution of the third and fourth term will be

$$T \sum_{N < n \leqslant T} n^{-2} + O(\log^{-2} T).$$

Therefore (23) gives

$$R(T) = T \sum_{1 \leqslant n \leqslant T} n^{-2} - \zeta(2)T + O(\log N) + O(\log^{-2} T)$$
$$= \zeta(2)T - \zeta(2)T + O(\log N) = O(\log^{2/3} T(\log\log T)^{1/3}),$$

which proves (4). With a little more effort one could presumably remove the log log – factor in (4).

To prove (5) it suffices to prove

$$(24) \quad \int_T^{2T} R(t)dt = O(T),$$

and then to replace T by $2^{-j}T$ and sum over $j = 1, 2, \ldots$. For this purpose the error term in (15) is too large. Thus we use first (8), (10) and (14) to write, for $T \leqslant t \leqslant 2T$,

$$R(t) = \int_0^t \left(| \sum_{n \leqslant T} n^{-1-iu} |^2 - \zeta(2) \right) du - 2\mathrm{Re} \left\{ \int_1^t \sum_{n \leqslant T} \left(\frac{t}{n}\right)^{iu} \frac{du}{inu} \right\}$$
$$+ O(1),$$

which is more precise than (15). We square out the first sum above and then integrate termwise, noting that interchanging m and n we have

$$\Sigma := \sum_{1 \leqslant m \neq n \leqslant T} \frac{1}{imn \log(m/n)} = \sum_{1 \leqslant m \neq n \leqslant T} \frac{1}{inm \log(n/m)} = -\Sigma,$$

hence $\Sigma = 0$. This gives, for $T \leqslant t \leqslant 2T$,

$$(25)\quad R(t) = \sum_{1 \leqslant m \neq n \leqslant T} \frac{(m/n)^{it}}{imn \log(m/n)} - 2\,\mathrm{Re}\left\{\frac{1}{i} \int_1^t \sum_{n \leqslant T} \left(\frac{t}{n}\right)^{iu} \frac{du}{un}\right\} + O(1)\,.$$

Then we integrate (25) to obtain

$$\int_T^{2T} R(t)\,dt = \sum_{1 \leqslant m \neq n \leqslant T} \frac{(m/n)^{iT} - (m/n)^{2iT}}{mn \log^2(m/n)}$$

$$- 2\,\mathrm{Re}\left\{\frac{1}{i} \int_T^{2T} \int_1^t \sum_{n \leqslant T} \left(\frac{t}{n}\right)^{iu} \frac{du\,dt}{nu}\right\} + O(T)\,.$$

But the modulus of the double sum above does not exceed

$$\sum_{1 \leqslant n < m \leqslant T} \frac{4}{mn \log^2(m/n)} = \sum_{1 \leqslant n < m \leqslant T,\, n < m/2} + \sum_{1 \leqslant n < m \leqslant T,\, n \geqslant m/2}$$

$$\ll \log^2 T + \sum_{n \leqslant T} \sum_{n < m \leqslant 2n} \frac{1}{m^2 \log^2(m/n)} \ll \log^2 T + \sum_{n \leqslant T} \sum_{n < m \leqslant 2n} \frac{1}{(m-n)^2}$$

$$\ll T\,,$$

and it remains to estimate

$$I := \int_T^{2T} \int_1^t \sum_{n \leqslant T} \left(\frac{t}{n}\right)^{iu} \frac{du\,dt}{nu} = \int_T^{2T} \int_1^{\log T} \sum_{n \leqslant T/2} \left(\frac{t}{n}\right)^{iu} \frac{du\,dt}{nu}$$

$$+ \int_T^{2T} \int_{\log T}^t \sum_{n \leqslant T/2} \left(\frac{t}{n}\right)^{iu} \frac{du\,dt}{nu} + \int_T^{2T} \int_1^t \sum_{T/2 < n \leqslant T} \left(\frac{t}{n}\right)^{iu} \frac{du\,dt}{nu}$$

$$= I_1 + I_2 + I_3\,,$$

say. We have

$$I_1 = \sum_{n \leqslant T/2} \frac{1}{n} \int_1^{\log T} \left(\int_T^{2T} \left(\frac{t}{n}\right)^{iu} \frac{du}{u}\right) dt$$

$$= \sum_{n \leqslant T/2} \frac{1}{n} \int_1^{\log T} \left\{\frac{(t/n)^{iu}}{iu \log(t/n)}\Bigg|_T^{2T} + \int_T^{2T} \frac{(t/n)^{iu}}{iu^2 \log(t/n)}\,du\right\} dt$$

$$\ll \frac{\log T}{T} \sum_{n \leqslant T/2} \frac{1}{n \log(T/n)} \ll \frac{\log^2 T}{T}\,,$$

$$I_2 = \sum_{n \leqslant T/2} \frac{1}{n} \int_T^{2T} \left\{ \frac{(t/n)^{iu}}{iu \log (t/n)} \Bigg|_{\log T}^{t} + \int_{\log T}^{t} \frac{(t/n)^{iu}}{iu^2 \log (t/n)} \, du \right\} dt$$

$$\ll \sum_{n \leqslant T/2} \frac{T}{n \log (T/n) \log T} \ll T ,$$

$$I_3 = \sum_{T/2 < n \leqslant T} \frac{1}{n} \int_1^{2T} \left(\int_{\max(u,T)}^{2T} \left(\frac{t}{n}\right)^{iu} dt \right) \frac{du}{u}$$

$$= \sum_{T/2 < n \leqslant T} \int_1^{2T} n^{-1-iu} \frac{(2T)^{iu+1} - (\max(u,T))^{iu+1}}{u(iu+1)} \, du$$

$$\ll T \sum_{T/2 < n \leqslant T} \frac{1}{n} \ll T ,$$

and (24) follows.

It remains yet to prove (6), which follows from

$$\int_T^{2T} R^2(t) dt = O\big(T (\log \log T)^4\big) . \tag{26}$$

We use (25) and (11) to obtain

$$R^2(t) \leqslant \sum_{a \neq b \leqslant T,\, c \neq d \leqslant T} \frac{-2 \left(\dfrac{ad}{bc}\right)^{it}}{abcd \log (a/b) \log (d/c)} + O\big((\log \log T)^2\big)$$

$$= \sum_{ad = bc} + \sum_{0 < |\log(ad/bc)| \leqslant T^{-1} \log^4 T} + \sum_{|\log(ad/bc)| > T^{-1} \log^4 T}$$

$$+ O\big((\log \log T)^2\big) = S_0 + S_1 + S_2 + O\big((\log \log T)^2\big) ,$$

say. On integrating we obtain

$$\int_T^{2T} S_2 dt$$

$$= 2 \sum_{a \neq b \leqslant T,\, c \neq d \leqslant T,\, |\log(ad/bc)| > T^{-1} \log^4 T} \frac{\left(\dfrac{ad}{bc}\right)^{iT} - \left(\dfrac{ad}{bc}\right)^{2iT}}{iabcd \log \left(\dfrac{a}{b}\right) \log \left(\dfrac{d}{c}\right) \log \left(\dfrac{ad}{bc}\right)}$$

$$\ll \frac{T}{\log^4 T} \left(\sum_{a \neq b \leqslant T} \frac{1}{ab \, |\log (a/b)|} \right)^2 \ll T ,$$

since an elementary argument easily gives

$$\sum_{a \neq b \leqslant T} \frac{1}{ab \,|\log(a/b)|} \ll \log^2 T\,.$$

The remaining sums S_0 and S_1 are not integrated, but it is sufficient to show that

$$S_0 \ll 1, S_1 \ll (\log\log T)^4\,. \tag{27}$$

We have

$$S_0 = \sum_{ad = bc,\, a \neq b \leqslant T,\, c \neq d \leqslant T} \frac{1}{abcd \log(a/b) \log(d/c)}$$

$$= \sum_k \frac{1}{k^2} \sum_{ad = bc = k,\, a \neq b \leqslant T} \frac{1}{\log^2(a/b)} = \sum_{a \neq b \leqslant T} \frac{1}{\log^2(a/b)} \sum_{a|k,\, b|k} \frac{1}{k^2}$$

$$\ll \sum_{a \neq b \leqslant T} \frac{1}{[a,b]^2 \log^2(a/b)} \leqslant \sum_{j=1}^{\infty} \sum_{a \neq b,\, (a,b) = j} \frac{1}{\log^2(a/b)} \left(\frac{j}{ab}\right)^2$$

$$\leqslant \sum_{j=1}^{\infty} \frac{1}{j^2} \sum_{a' \geqslant 1,\, b' \geqslant 1,\, a' \neq b'} \frac{1}{\log^2(a'/b')\,(a'b')^2} = O(1)\,.$$

The proof of the other bound in (27) is also elementary, but somewhat more involved. Write

$$S_1 = S_3 + S_4\,,$$

where in S_3 summation is over a, b, c, d such that $1 \leqslant a \neq b \leqslant T$,

$$1 \leqslant c \neq d \leqslant T, |\log a/b| \geqslant T^{-1/4}, |\log c/d| \geqslant T^{-1/4}\,,$$

$$0 < \left|\log \frac{ad}{bc}\right| \leqslant T^{-1} \log^4 T\,,$$

and in S_4 over the remaining values of a, b, c, d. Thus

$$S_3 \ll T^{1/2} \sum_{a,b,c,d \leqslant T;\, |\log(ad/bc)| \leqslant T^{-1}\log^4 T} \frac{1}{abcd}$$

$$\ll T^{1/2} \sum_{|\log(k/l)| \leqslant T^{-1}\log^4 T;\, k \neq l \leqslant T^2} \frac{d(k)d(l)}{kl}\,.$$

Now observe that if $k \neq l \geqslant 1$ are integers, then

$$\left|\log \frac{k}{l}\right| \geqslant \log \frac{l+1}{l} \geqslant \frac{1}{2l}\,, \tag{28}$$

so that

$$l \geqslant \frac{1}{2} T \log^{-4} T, \quad \frac{|k-l|}{l} \ll \left| \log\left(1 + \frac{k-l}{l}\right) \right| \leqslant T^{-1} \log^4 T .$$

Thus

$$S_3 \ll T^{1/2+\varepsilon} \sum_{T/(2\log^4 T) \leqslant l \leqslant T^2} \frac{1}{l^2} \left(\frac{\log^4 T}{T} + 1 \right) \ll T^{-1/2+2\varepsilon} \ll 1$$

for $0 < \varepsilon \leqslant 1/4$. In S_4 we have either $|\log(a/b)| \leqslant T^{-1/4}$ or $|\log(d/c)| \leqslant T^{-1/4}$. If the former holds, then

$$\left| \log \frac{d}{c} \right| = \left| \log \frac{ad}{bc} - \log \frac{a}{b} \right| \leqslant 2T^{-1/4} ,$$

so that in S_4 we have $|\log(a/b)| \leqslant 2T^{-1/4}$, $|\log(d/c)| \leqslant 2T^{-1/4}$, and also $a \ll b \ll a, c \ll d \ll c$. Setting $a = b + j_1, c = d + j_2$, we have that $j_1 \ll bT^{-1/4}, j_2 \ll dT^{-1/4}$, and

$$\left| \frac{j_1}{b} - \frac{j_2}{d} \right| \ll \frac{\log^4 T}{T} , \tag{29}$$

since

$$\left| \frac{a}{b} - \frac{c}{d} \right| = \frac{|ad-bc|}{bd} \ll \frac{|ad-bc|}{bc} \ll \left| \log\left(1 + \frac{ad-bc}{bc}\right) \right|$$

$$= \left| \log \frac{ad}{bc} \right| \leqslant \frac{\log^4 T}{T} .$$

Hence

$$S_4 \ll \sum_{b,d,j_1,j_2}^{*} \frac{1}{bdj_1j_2} ,$$

where $\sum^*$ denotes summation with the conditions $b, d \leqslant T; j_1 \ll bT^{-1/4}$, $j_2 \ll dT^{-1/4}$ and (29) satisfied. We have

$$S_4 = S_5 + S_6 ,$$

where trivially

$$S_5 = \sum_{T\log^{-10} T \leqslant b, d, \leqslant T; j_1, j_2 \leqslant \log^{20} T} \frac{1}{bdj_1j_2} \ll (\log\log T)^4 ,$$

and in S_6 we have (by symmetry) either

i) $j_2 \geqslant \log^{20} T$

or

ii) $d \leqslant T \log^{-10} T$.

From (29) it follows that

$$d - \frac{bj_2}{j_1} \ll \frac{bd \log^4 T}{j_1 T}. \tag{30}$$

Suppose that i) holds. Then $d - bj_2 j_1^{-1} \ll j_1^{-1} b \log^4 T$ from (30), hence the corresponding part of S_6 is

$$\ll \sum_{b \leqslant T} \frac{1}{b} \sum_{j_1 \ll bT^{-1/4}} \frac{1}{j_1} \sum_{j_2 \geqslant \log^{20} T} \frac{1}{j_2} \sum_{|d - bj_2 j_1^{-1}| \leqslant Cbj_1^{-1} \log^4 T} \frac{1}{d}$$

$$\ll \sum_{b \leqslant T} \frac{1}{b} \sum_{j_1 \ll bT^{-1/4}} \frac{1}{j_1} \sum_{j_2 \geqslant \log^{20} T} j_2^{-2} \log^4 T \ll \log^{-4} T.$$

If ii) holds, then (30) gives

$$d - \frac{bj_2}{j_1} \ll \frac{b}{j_1} \log^{-6} T,$$

and the corresponding contribution to S_6 will be again $\ll \log^{-4} T$. This proves (27) and completes the proof of the Theorem.