Commutative algebra

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Fischer. He, too, was a specialist in invariant theory, but invariant theory of the Hilbert persuasion. Emmy Noether came under his influence and gradually made the change from Gordan's algorithmic approach to invariant theory to Hilbert's conceptual approach. Later work on invariants brought her in contact with the famous joint paper of Dedekind and Weber (see p. 115 below) on the arithmetic theory of algebraic functions. She became "sold" on Dedekind's approach and ideas, and this determined the direction of her future work.

COMMUTATIVE ALGEBRA

The two major sources of commutative algebra are algebraic geometry and algebraic number theory. Emmy Noether's two seminal papers of 1921 and 1927 on the subject can be traced, respectively, to these two sources. In these papers, entitled, respectively, Ideal Theory in Rings (Idealtheorie in Ringbereichen) and Abstract Development of Ideal Theory in Algebraic Number Fields and Function Fields (Abstrakter Aufbau der Idealtheorie in algebraischen Zahl-und Funktionenkörpern), she broke fundamentally new ground, originating "a new and epoch-making style of thinking in algebra" ([41], p. 130).

Algebraic geometry had its origins in the study, begun in the early 19th century, of abelian functions and their integrals. This analytic approach to the subject gradually gave way to geometric, algebraic, and arithmetic means of attack. In the algebraic context, the main object of study is the ring of polynomials $k[x_1, x_2, \ldots, x_n]$, k a field (in the 19th century k was the field of real or complex numbers). Hilbert in the 19th century, and Lasker and Macauley in the early 20th century, had shown that in such a ring every ideal is a finite intersection of primary ideals, with certain uniqueness properties.[1]) (Geometrically, the result says that every variety is a unique, finite, union of irreducible varieties.) In her 1921 paper Emmy Noether generalized this result to arbitrary commutative rings with the ascending chain condition (a.c.c.).[2]) Her main result was that in such a ring every ideal is a finite intersection (with accompanying uniqueness properties) of primary ideals. (See [14] for historical and [3] for technical details.)

What was so significant about this paper which (we recall) MacLane singled out as marking the beginning of abstract algebra as a conscious discipline?

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[1]) An ideal I in a commutative ring R is called *primary* if $xy \in I$ implies $x \in I$ or $y^t \in I$ for some positive integer t. The concept of primary ideal is an extension to rings of prime power for the integers.

[2]) A commutative ring R satisfies the *ascending chain condition* if every ascending chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ terminates; i.e., $I_n = I_{n+1} = \ldots$ for some positive integer n.

First and foremost was the isolation of the a.c.c. as the crucial concept needed in the proof of the main result. In fact, the proof "rested entirely on elementary consequences of the chain condition and... [was] startling in... simplicity" ([22], p. 13). Earlier proofs (of the corresponding result for polynomial rings) involved considerable computation, such as elimination theory and the geometry of algebraic sets.

The a.c.c. did not originate with Emmy Noether. Dedekind (in 1894) and Lasker (in 1905) used it, but in concrete settings of rings of algebraic integers and of polynomials, respectively. Moreover, the a.c.c. was for them incidental rather than of major consequence. Noether's isolation of the a.c.c. as an important concept was a watershed. Thanks to her work, rings with the a.c.c., now called noetherian rings [1]), have been singled out for special attention. In fact, commutative algebra has been described as the study of (commutative) noetherian rings. As such, the subject had its formal genesis in Emmy Noether's 1921 paper.

Another fundamental concept with Emmy Noether highlighted in the 1921 paper is that of a ring. This concept, too, did not originate with her. Dedekind (in 1871) introduced it as a subset of the complex numbers closed under addition, substraction, and multiplication, and called it an "order". Hilbert (in 1897), in his famous Report on Number Theory (Zahlbericht), coined the term "ring", but only in the context of rings of integers of algebraic number fields. Fraenkel (in 1914) gave essentially the modern definition of ring, but postulated two extraneous conditions. Noether (in the 1921 paper) gave the definition in current use (given also, apparently, by Sono in 1917, but this went unnoticed).

But it was not merely Noether's *definition* of the concept of ring which proved important. Through her groundbreaking papers in which the concept of ring played an essential role (and of which the 1921 paper was an important first), she brought this concept into prominence as a central notion of algebra. It immediately began to serve as the starting point for much of abstract algebra, taking its rightful place alongside the concepts of group and field, already reasonably well established at that time.

Noether also began to develop in the 1921 paper a general theory of ideals for commutative rings. Notions of prime, primary, and irreducible ideal, of intersection and product of ideals, of congruence modulo an ideal — in short, much of the machinery of ideal theory, appears here.

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[1]) A term coined in 1943 by Chevalley.

Toward the end of the paper she defined the concept of *module* over a noncommutative ring and showed that some of the earlier decomposition results for ideals carry over to submodules. (I will discuss modules in connection with Noether's work in noncommutative algebra.)

To summarize, the 1921 paper introduced and gave prominence to what came to be some of the basic concepts of abstract algebra, namely ring, module, ideal, and the a.c.c. Beyond that, it introduced, and began to show the efficacy of, a new way of doing algebra — abstract, axiomatic, conceptual. No mean accomplishment for a single paper! (See [19] and [22] for further details.)

Emmy Noether's 1927 paper had its roots in algebraic number theory and, to a lesser extent, in algebraic geometry. The sources of algebraic number theory are Gauss' theory of quadratic forms of 1801, his study of biquadratic reciprocity of 1832 (in which he introduced the Gaussian integers), and attempts in the early 19th century to prove Fermat's Last Theorem. In all cases the central issue turned out to be unique factorization in rings of integers of algebraic number fields.[1]) When examples of such rings were found in which unique factorization fails,[2]) the problem became to try to "restore", in some sense, the "paradise lost". This was achieved by Dedekind in 1871 (and, in a different way, by Kronecker in 1882) when he showed that unique factorization can be reestablished if one considers factorization of ideals (which he had introduced for this purpose) rather than of elements. His main result was that if R is the ring of integers of an algebraic number field, then every ideal of R is a unique product of prime ideals.[3]) (See [6] for historical and [34] for technical details.)

Riemann introduced "Riemann surfaces" in the 1850s in order to facilitate the study of (multivalued) algebraic functions. His methods were, however, nonrigorous, and depended on physical considerations. In 1882 Dedekind and Weber wrote an all-important paper whose aim was to give rigorous, algebraic, expression to some of Riemann's ideas on complex

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[1]) An *algebraic number field* is a finite extension of the rationals, $Q(\alpha) = \{a_0 + a_1\alpha + \ldots + a_n\alpha^n : a_i \in Q, \alpha \text{ an algebraic number}\}$. The *ring of integers* of $Q(\alpha)$ consists of the elements of $Q(\alpha)$ which are roots of *monic* polynomials with *integer* coefficients. See [1] for details.

[2]) $R = \{a + b\sqrt{-5} : a, b \in \mathbf{Z}\}$ is such an example. Here

$$6 = 2 \times 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

are two distinct decomposition of 6 as a product of primes of R.

[3]) An ideal I of a ring R is said to be *prime* if $xy \in I$ implies $x \in I$ or $y \in I$. Prime ideals are generalizations of primes in the ring of integers.

function theory, in particular to his notion of a Riemann surface. Their idea was to establish an analogy between algebraic number fields and algebraic function fields, and to carry over the machinery and results of the former to the latter. They succeeded admirably, giving (among other things) a purely algebraic definition of a Riemann surface, and an algebraic proof of the fundamental Riemann-Roch Theorem. At least as importantly, they pointed to what proved to be a most fruitful idea, namely the interplay between algebraic number theory and algebraic geometry.

More specifically, just as in algebraic number theory one associates an algebraic number field $Q(\alpha)$ with a given algebraic number, so in algebraic geometry one associates an algebraic function field $\mathbf{C}(x, y)$ with a given algebraic function. $\mathbf{C}(x, y)$ consists of polynomials in x and y with complex coefficients, where y satisfies a polynomial equation with coefficients in $\mathbf{C}(x)$ (i.e., y is algebraic over $\mathbf{C}(x)$).[1] If A is the "ring of integers" of $\mathbf{C}(x, y)$ (i.e., A consists of the roots in $\mathbf{C}(x, y)$ of *monic* polynomials with coefficients in $\mathbf{C}[x]$), then a major result of the Dedekind-Weber paper is that every ideal in A is a unique product of prime ideals. (See [14] and [26] for historical, and [9] and [16] for technical, details.)

In her 1927 paper Emmy Noether generalized the above decomposition results for algebraic number fields and function fields to commutative rings. In fact, she characterized those commutative rings in which every ideal is a unique product of prime ideals. Such rings are now called *Dedekind domains.* She showed that R is a Dedekind domain if and only if (1) R statisfies the a.c.c., (2) R/I statisfies the d.c.c. for every nonzero ideal I of R, (3) R is an integral domain (i.e., it has an identity and no zero divisors), and (4) R is integrally closed in its field of quotients. Condition (4) proved particularly significant since it singled out the basic notion of integral dependence (related to that of integral closure).[2]) This concept (already present in Dedekind's work on algebraic numbers) has proved to be of fundamental importance in commutative algebra. As Gilmer notes, "the concept of integral dependence is to *Aufbau* [Noether's 1927 paper] what the a.c.c. is to *Idealtheorie* [her 1921 paper]" ([19], p. 136). Among other basic results she proved in this paper are: (a) the (by now standard) isomorphism and homomorphism theorems for rings and modules, (b) that a module M has a composition series if and only if it

[1]) $\mathbf{C}(x, y)$ is an extension field of $\mathbf{C}$ of transcendence degree 1; i.e., x is transcendental over $\mathbf{C}$ and y is algebraic over $\mathbf{C}(x)$. Thus, in analogy with the algebraic number field $Q(\alpha)$, $\mathbf{C}(x)$ corresponds to Q and y to α.

[2]) Let $R \subseteq S$ be rings. An element $s \in S$ is *integrally dependent* on R (or is integral over R) if it satisfies a monic polynomial with coefficients in R. R is *integrally closed* in S if every element of S which is integral over R belongs to R.

satisfies both the a.c.c. and d.c.c., (c) that if an R-module M is finitely generated and R satisfies the a.c.c. (d.c.c.), then so does M.

To summarize Emmy Noether's contributions to commutative algebra: in addition to proving important results, she introduced concepts and developed techniques which have become standard tools of the subject. In fact, her 1921 and 1927 papers, combined with those of Krull of the 1920s, are said to have created the subject of commutative algebra.

Noncommutative algebra and representation theory

Before her ideas in commutative algebra had been fully assimilated by her contemporaries, Emmy Noether turned her attention to the other major algebraic subjects of the 19th and early 20th centuries, namely hypercomplex number systems (what we now call associative algebras) and groups (in particular, group representations). She extended and unified these two subjects through her abstract, conceptual approach, in which module-theoretic ideas that she had used in the commutative case played a crucial role.

The theory of hypercomplex systems began with Hamilton's 1843 introduction of the quaternions. At the end of the 19th century, E. Cartan, Frobenius, and Molien gave structure theorems for such systems over the real and complex numbers, and in 1907 Wedderburn extended these to hypercomplex systems over arbitrary fields. In the spirit of Emmy Noether's work in commutative algebra, Artin extended Wedderburn's results to (noncommutative, semisimple) rings with the descending chain condition. (See [25] for details.)

Groups were the first algebraic systems to be developed extensively. By the end of the 19th century they began to be studied abstractly. An important tool in that study was representation theory, developed by Burnside, Frobenius, and Molien in the 1890s (see [20]). The idea was to study, instead of the abstract group, its concrete representations in terms of matrices (A *representation* of a group is a homomorphism of the group into the group of invertible matrices of some given order.)

In her 1929 paper Hypercomplex Numbers and Representation Theory (Hyperkomplexe Grössen und Darstellungstheorie) Emmy Noether framed group representation theory in terms of the structure theory of hypercomplex systems. The main tool in this approach was the *module.* The idea was to associate with each representation ϕ of G by invertible matrices with entries in some field k, a $k(G)$-module V called the *representation module* of ϕ($k(G)$ is the *group algebra* of G over k). Conversely, any $k(G)$-module M gives rise