Applications of noncommutative to commutative algebra

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spaces, and many of the standard vector-space constructions, such as subspace, quotient space, direct sum, and tensor product carry over to modules.)[1]) In fact, the importance of the invention of *homological algebra* was that it carried the process of linearization far forward by developing tools for its implementation. (E.g., the functors "Ext" and "Tor" measure the extent to which modules over general rings "misbehave" when compared to modules over fields, viz. vector spaces; see [8].)

Applications of noncommutative to commutative algebra

Noether believed that the theory of noncommutative algebras is governed by simpler laws than that of commutative algebra. In her 1932 plenary address at the International Congress of Mathematicians in Zurich, entitled Hypercomplex Systems and their Relations to Commutative Algebra and Number Theory (Hyperkomplexe Systeme in ihren Beziehungen zur kommutativen Algebra und Zahlentheorie), she outlined a program putting that belief into practice. Her program has been called "a foreshadowing of modern cohomology theory" ([35], p. 8). The ideas on factor sets contained therein were soon used by Hasse and Chevalley "to obtain some of the main results on global and local class field theory" ([22], p. 26). Noether's own immediate objective was to apply the theory of central simple algebras (as developed by her, Brauer, and others) to problems in class field theory. (See [7], [35], and [36].)

Some of her ideas (and those of others) on the interplay between commutative and noncommutative algebra had already recently born fruit with the proof of the celebrated Albert-Brauer-Hasse-Noether Theorem. This result, called by Jacobson "one of the high points of the theory of algebras" ([22], p. 21), gives a complete description of finite-dimensional division algebras over algebraic number fields.[2]) It is important in the study of finite-dimensional algebras and of group representations.

To bring out the context of the above theorem, it should be noted that Wedderburn's 1907 structure theorems for finite-dimensional algebras reduced their study to that of nilpotent algebras and division algebras. Since the unravelling of the structure of the former seemed (and still seems, despite considerable progress) "hopeless", attention focussed on the latter.

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[1]) We know the power of linearization in analysis. Modules can be said to provide analogous power in algebra.

[2]) They are intimately related to the "cyclic" algebras studied earlier by Dickson (see [21], Vol. II, p. 480).

Considerable progress on the structure of division algebras was made in the late 1920s and early 1930s. The Albert-Brauer-Hasse-Noether Theorem was a high point of these researches. It should be stressed, however, that even today much is still unknown about finite-dimensional division algebras.

C. Her legacy

The concepts Emmy Noether introduced, the results she obtained, and the mode of thinking she promoted, have become part of our mathematical culture. As Alexandrov put it ([2], p. 158):

> It was she who taught us to think in terms of simple and general algebraic concepts — homomorphic mappings, groups and rings with operators, ideals — and not in cumbersome algebraic computations; and [she] thereby opened up the path to finding algebraic principles in places where such principles had been obscured by some complicated special situation...

Moreover, as Weyl noted, "her significance for algebra cannot be read entirely from her own papers; she had great stimulating power and many of her suggestions took shape only in the works of her pupils or co-workers" ([41], pp. 129-130). Indeed, Weyl himself acknowledged his indebtedness to her in his work on groups and quantum mechanics. Among others who have *explicitly* mentioned her influence on their algebraic works are Artin, Deuring, Hasse, Jacobson, Krull, and Kurosh.

Another important vehicle for the spread of Emmy Noether's ideas was the now-classic treatise of van der Waerden entitled "Modern Algebra", first published in 1930. (It was based on lectures of Noether and Artin — see [39].) Its wealth of beautiful and powerful ideas, brilliantly presented by van der Waerden, has nurtured a generation of mathematicians. The book's immediate impact is poignantly described by Dieudonné and G. Birkhoff, respectively:

> I was working on my thesis at that time; it was 1930 and I was in Berlin. I still remember the day that van der Waerden came out on sale. My ignorance in algebra was such that nowadays I would be refused admittance to a university. I rushed to those volumes and was stupefied to see the new world which opened before me. At that time my knowledge of algebra went no further than *mathématiques spéciales,* determinants, and a little on the solvability of equations and unicursal curves. I had graduated from the École Normale and I did not know what an ideal was, and only just knew what a group was! This gives you an idea of what a young French mathematician knew in 1930 ([13], p. 137).