2. Numbers of Constant Type

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **38 (1992)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

If x is rational, then its continued fraction expansion terminates, and we can write $x = [a_0, a_1, \ldots, a_n]$. If we agree that $a_n = 1$ and $n \geqslant 1$, then this expansion is unique and we define

$$K(x) = \max_{1 \leqslant k \leqslant n} a_k ,$$

the largest partial quotient in the continued fraction for x.

If x is irrational, then its continued fraction expansion does not terminate. This expansion is unique. We write $x = [a_0, a_1, a_2, \ldots]$ and define

$$K(x) = \sup_{k \geqslant 1} a_k .$$

If $K(x) < \infty$, then we say that x has *bounded partial quotients.*

We define $\mathscr{B}_k = \{x \in \mathbf{R} \mid K(x) \leqslant k\}$, and $\mathscr{B} = \{x \in \mathbf{R} \mid K(x) < \infty\}$. Furthermore, let $\mathscr{E}_k = \mathscr{B}_k \cap (0, 1)$ and $\mathscr{E} = \mathscr{B} \cap (0, 1)$.

Real numbers with bounded partial quotients appear in many fields of mathematics and computer science: Diophantine approximation, fractal geometry, transcendental number theory, ergodic theory, numerical analysis, pseudo-random number generation, dynamical systems, and formal language theory. In this paper we survey some of these applications. Because of limited space, we cannot include a discussion of every result in detail. However, we have tried to include as complete a list of references as possible for those topics directly related to the main subject. Readers who know of other references are urged to contact the author (and provide a copy of the relevant paper, if possible). It is hoped that the list of references may contain some surprises even for experts in the field.

The author's interest in the subject arose from the material in Section 9. Because of this, the viewpoint presented in this article may be somewhat idiosyncratic.

2. Numbers of Constant Type

Let θ be an irrational number, and let $\| \theta \|$ denote the distance between θ and the closest integer.

Let $r \geqslant 1$ be a real number. We say that θ is *of type* $< r$ if

$$q \| q\theta \| \geqslant \frac{1}{r}$$

for all integers $q \geqslant 0$. Then we have the following

THEOREM 1. *If* θ *is of type* $< r$, *then* $K(\theta) < r$. *If* $K(\theta) = r$, *then* θ *is of type* $< r + 2$.

For a proof, see Baker [20, p. 47] or Schmidt [272, p. 22].

If there exists an $r < \infty$ such that θ is of type $< r$, then θ is said to be of *constant type.* By the theorem, numbers of constant type and numbers with bounded partial quotients coincide, and we will use these terms interchangeably in what follows.

A classical theorem of Lagrange states that the continued fraction for x is ultimately periodic if and only if x is a real quadratic irrational, and so all real quadratic irrationals are of constant type; see, for example, Lagrange [178] or Hardy and Wright [135, Chapter 10]. We will not explicitly discuss quadratic irrationals further in this paper.

Since

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \ldots]$$

(see Cotes [58] and Euler [102]), we see that e is *not* of constant type. It is also known that the numbers $e^{2/n}$ and $\tan 1/n$ (n an integer $\geqslant 1$) are not of constant type. The status of π and γ (Euler's constant) is presently unknown. In section 9 we will see some explicit examples of transcendental numbers of constant type.

One way to interpret Theorem 1 is to say that numbers with bounded partial quotients are *badly approximable* by rationals; this term is also used frequently in the literature.

Note that if θ is of constant type, and r is a rational number, then $r\theta$ is also of constant type [54]. In fact, it is not hard to prove the following: let $r = a/b$ be a rational number, and suppose $K(\theta) = n$. Then $K(r\theta) \leqslant |ab|(n+2)$, and $K(\theta + r) \leqslant b^2(n+2)$; see Cusick and Mendès France [69].

From this, it follows that if a, b, c, d are integers with $ad - bc \neq 0$, then

$$\frac{a\theta + b}{c\theta + d}$$

has bounded partial quotients iff θ does. (See Shallit [278]. I would like to thank J. C. Lagarias for bringing this to my attention.) One can also deduce this result directly from the continued fraction, using results of Raney [256]. For another view of Raney's results, see van der Poorten [246].

Another related concept is the *Lagrange-Markoff* constant, denoted by $\mu(\theta)$. It is defined as follows:

$$\mu(\theta)^{-1} = \liminf_{q \to \infty} q \, \| q\theta \| .$$

Hurwitz [150] showed, among other things, that $\mu(\theta) \geqslant \sqrt{5}$; furthermore, $\mu\left(\frac{1+\sqrt{5}}{2}\right) = \sqrt{5}$. Perron [234] showed that if

$$\theta = [a_0, a_1, a_2, \ldots] ,$$

then

$$\mu(\theta) = \limsup_{i \to \infty} ([a_{i+1}, a_{i+2}, a_{i+3}, \ldots] + [0, a_i, a_{i-1}, \ldots, a_1]) .$$

From this it follows that $\mu(\theta) < \infty$ if and only if θ is of constant type.

The range of $\mu(\theta)$, as θ takes on all irrational values, is known as the *Lagrange spectrum* and has been extensively studied. We direct the reader to the work of Lagrange [178, pp. 26-27]; Markoff [203, 204]; Heawood [138]; Perron [235]; Vinogradov, Delone, and Fuks [295]; Freiman [111]; Kinney and Pitcher [166]; Beršstein [29]; Davis and Kinney [78]; Cusick [59, 62]; Flahive [117]; Cusick and Mendès France [69]; Wilson [301]; Dietz [89]; Pavone [232]; Prasad [249]; and especially the books of Koksma [172] and Cusick and Flahive [67].

For more on approximation by rational numbers, see Cassels [52], Schmidt [272], Kraaikamp and Liardet [313], Larcher [312].

3. The Metric Theory of Continued Fractions

Recall that $\mathscr{C}$ denotes the set of real numbers in (0, 1) with bounded partial quotients.

While it is easy to see $\mathscr{C}$ has uncountably many elements, nevertheless "most" numbers do *not* have bounded partial quotients. More precisely, we have the following

THEOREM 2 (Borel-Bernstein). *$\mathscr{C}$ is a set of measure* 0.

The theorem is due to Borel [38]. The original proof was not complete, as discussed in Bernstein [27]; further details were provided in a later paper of Borel [39]. For other proofs, see Hardy and Wright [135, Thm. 196] or Khintchine [160]. Also see Dyson [96].

Here is a sketch of a more general theorem: first, let us equate probability with Lebesgue measure, and assume x is a real number in (0, 1). Then, expanding x as a continued fraction, we have

$$x = [0, a_1, a_2, a_3, \ldots] ,$$