6. Fractal Geometry

They showed that if θ has bounded partial quotients, then $U_n(\theta) = O(\log n)$, $V_n(\theta) = O(n)$, and $W_n(\theta) = O(n^2)$. See [134].

(Warning to the reader: in their papers, Hardy and Littlewood used the notation $\{x\}$ to mean $x - [x] - \frac{1}{2}$, not $x - [x]$, as is more standard today.)

For other related papers, see Hardy and Littlewood [129, 131]; the collected works of Hardy [127]; Ostrowski [230]; Khintchine [161]; Oppenheim [228]; Chowla [53, 54]; Walfisz [298, 299, 300], and Schoissengeier [314].

Others researchers have examined similar sums in connection with numbers with bounded partial quotients. See the papers of Faĭziev [104] Ivanov [151], and Schoissengeier [274].

6. Fractal Geometry

Numbers with bounded partial quotients provided an early example of a set with non-integral Hausdorff dimension.

Let $\dim S$ denote the Hausdorff dimension of the set S (for a definition, see, e.g. Falconer [103]). We use the definitions of $\mathscr{E}$ and $\mathscr{E}_k$ from section 1.

In 1928, Jarník [152] proved that $\dim \mathscr{E} = 1$,

$$\frac{1}{4} < \dim \mathscr{E}_2 < 1 ,$$

and

$$1 - \frac{4}{k \log 2} < \dim \mathscr{E}_k < 1 - \frac{1}{8k \log k} ,$$

for $k > 8$. An exposition of Jarník's work can be found in Rogers [263].

In 1941, Good proved the following result [118]:

$$\dim \mathscr{E}_k = \lim_{n \to \infty} \sigma_{k,n} ,$$

where $\sigma = \sigma_{k,n}$ is the real root of the equation

$$\sum_{1 \leqslant a_1, a_2, \ldots, a_n \leqslant k} Q(a_1, a_2, \ldots, a_n)^{-2\sigma} = 1$$

and $Q(\ )$ denotes Euler's continuant polynomial. (These are multivariate polynomials, defined by $Q(\ ) = 1$, $Q(a_1) = a_1$, and

$$Q(a_1, a_2, \ldots a_n) = a_n Q(a_1, a_2, \ldots, a_{n-1}) + Q(a_1, a_2, \ldots, a_{n-2})$$

for $n \geqslant 2$.)

Good also obtained the estimate $.5306 < \dim \mathscr{E}_2 < .5320$. This was improved by Bumby [48] in 1985 to $.5312 \leqslant \dim \mathscr{E}_2 \leqslant .5314$. More recently, Hensley [140] showed that $.53128049 < \dim \mathscr{E}_2 < .53128051$. For other results on the Hausdorff dimension of $\mathscr{E}_k$ and related sets, see Jarník [153]; Besicovitch [30]; Rogers [262]; Baker and Schmidt [21]; Hirst [147, 148]; Billingsley and Henningsen [32]; Cusick [63, 64, 65]; Pollington [245]; Kaufman [158]; Marion [202]; Gardner and Mauldin [115]; Ramharter [253, 254]; and Hensley [139, 141, 308, 309].

7. Schmidt's Game

W. M. Schmidt [270] introduced the following two-player game, called an (α, β) game: let α, β be real numbers with $0 < \alpha, \beta < 1$. First Bob chooses a closed interval on the real line, called B_1. Then Alice chooses a closed interval $A_1 \subset B_1$, such that the length of A_1 is α times the length of B_1. Then Bob chooses a closed interval $B_2 \subset A_1$, such that the length of B_2 is β times the length of A_1, and so on. If the intersection of all the intervals A_i is a number with bounded partial quotients, then Alice is declared the winner; otherwise Bob is declared the winner.

Schmidt showed that if $0 < \alpha < 1/2$, then Alice always has a winning strategy for this game. This is somewhat surprising, since as we have seen above, the set $\mathscr{E}$ of numbers with bounded partial quotients has Lebesgue measure 0.

Using the theory of (α, β) games, Schmidt also reproved the result of Jarník that $\mathscr{E}$ has Hausdorff dimension 1.

Several papers have proved other results on (α, β) games: see Schmidt [271]; Freiling [109, 110]; and Dani [70, 71, 72]. Also see Schmidt [272, Chapter 3].

8. Hall's theorem

If S and T are sets, then by $S + T$ we mean the set

$$\{s + t \mid s \in S, t \in T\} .$$

Similarly, by $S \cdot T$ we mean the set

$$\{st \mid s \in S, t \in T\} .$$

If S is a set of Lebesgue measure zero, then it is quite possible for $S + S$ to have positive measure. For example, if C denotes the Cantor set (numbers