8. Hall's theorem

Good also obtained the estimate $.5306 < \dim \mathscr{E}_2 < .5320$. This was improved by Bumby [48] in 1985 to $.5312 \leqslant \dim \mathscr{E}_2 \leqslant .5314$. More recently, Hensley [140] showed that $.53128049 < \dim \mathscr{E}_2 < .53128051$. For other results on the Hausdorff dimension of $\mathscr{E}_k$ and related sets, see Jarník [153]; Besicovitch [30]; Rogers [262]; Baker and Schmidt [21]; Hirst [147, 148]; Billingsley and Henningsen [32]; Cusick [63, 64, 65]; Pollington [245]; Kaufman [158]; Marion [202]; Gardner and Mauldin [115]; Ramharter [253, 254]; and Hensley [139, 141, 308, 309].

7. Schmidt's Game

W. M. Schmidt [270] introduced the following two-player game, called an (α, β) game: let α, β be real numbers with $0 < \alpha, \beta < 1$. First Bob chooses a closed interval on the real line, called B_1. Then Alice chooses a closed interval $A_1 \subset B_1$, such that the length of A_1 is α times the length of B_1. Then Bob chooses a closed interval $B_2 \subset A_1$, such that the length of B_2 is β times the length of A_1, and so on. If the intersection of all the intervals A_i is a number with bounded partial quotients, then Alice is declared the winner; otherwise Bob is declared the winner.

Schmidt showed that if $0 < \alpha < 1/2$, then Alice always has a winning strategy for this game. This is somewhat surprising, since as we have seen above, the set $\mathscr{E}$ of numbers with bounded partial quotients has Lebesgue measure 0.

Using the theory of (α, β) games, Schmidt also reproved the result of Jarník that $\mathscr{E}$ has Hausdorff dimension 1.

Several papers have proved other results on (α, β) games: see Schmidt [271]; Freiling [109, 110]; and Dani [70, 71, 72]. Also see Schmidt [272, Chapter 3].

8. Hall's theorem

If S and T are sets, then by $S + T$ we mean the set

$$\{s + t \mid s \in S, t \in T\} .$$

Similarly, by $S \cdot T$ we mean the set

$$\{st \mid s \in S, t \in T\} .$$

If S is a set of Lebesgue measure zero, then it is quite possible for $S + S$ to have positive measure. For example, if C denotes the Cantor set (numbers in [0, 1] containing only 0's and 2's in their ternary expansion), then C has measure 0, and it is not hard to show that $C + C = [0, 2]$; see Borel [40] or Pavone [233]. The result is due to Steinhaus [310]; I am most grateful to G. Myerson for bringing this to my attention.

As we have seen above, the set $\mathscr{B}$, and hence each $\mathscr{B}_k$, also has Lebesgue measure zero. In 1947 Hall proved the following theorem [126]:

THEOREM 3. *Every real number x can be written as $x = y + z$, where $y, z \in \mathscr{B}_4$. Every real number $x \geqslant 1$ can be written as $x = yz$, where $y, z \in \mathscr{B}_4$.*

An exposition of Hall's result can be found in Cusick and Flahive [67].

Using the notation of the first paragraph of this section, we could rephrase the statement of Hall's theorem as follows: $\mathscr{B}_4 + \mathscr{B}_4 = \mathbf{R}$, and $[1, \infty) \subseteq \mathscr{B}_4 \cdot \mathscr{B}_4$.

In 1973, Cusick [61] proved that $\mathscr{B}_3 + \mathscr{B}_3 + \mathscr{B}_3 = \mathbf{R}$, and $\mathscr{B}_2 + \mathscr{B}_2 + \mathscr{B}_2 + \mathscr{B}_2 = \mathbf{R}$. He also observed that $\mathscr{B}_3 + \mathscr{B}_3 \neq \mathbf{R}$, and $\mathscr{B}_2 + \mathscr{B}_2 + \mathscr{B}_2 \neq \mathbf{R}$. These results were independently discovered by Diviš [90] and J. Hlavka [1]) [149]. Hlavka also showed that $\mathscr{B}_3 + \mathscr{B}_4 = \mathbf{R}$, and similar results. Apparently the status of $\mathscr{B}_2 + \mathscr{B}_5$ and $\mathscr{B}_2 + \mathscr{B}_6$ is still open.

For results of a similar character, see Cusick [60]; Cusick and Lee [68]; and Bumby [47].

9. EXPLICIT EXAMPLES OF TRANSCENDENTAL NUMBERS WITH BOUNDED PARTIAL QUOTIENTS

In Lang [179] we find the following statement:

> No simple example of [irrational] numbers of constant type, other than the one given above [real quadratic irrationals], is known. The best guess is that there are no other "natural" examples.

(Also see Lang [180].)

However, in 1979 Kmošek [167] and Shallit [275] independently discovered the following "natural" example of numbers of constant type.

THEOREM 4. *Let $n \geqslant 2$ be an integer and define*

(1)
$$f(n) = \sum_{i \geqslant 0} n^{-2^i}.$$

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[1]) Note this is *not* same person as E. Hlawka!