2. The abstract algebras

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2. The abstract algebras

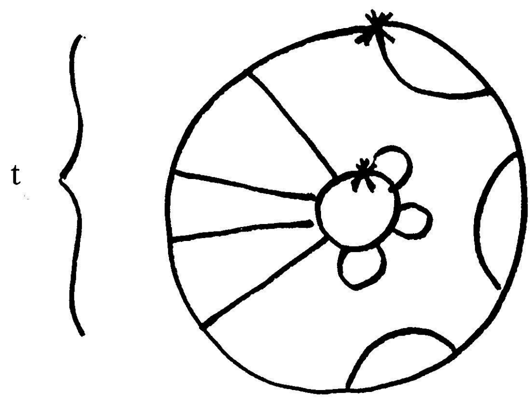
The (abstract) Brauer algebra with parameter $\delta \in \mathbf{C}$, $B(n, \delta)$, is the algebra with basis the set of all (n, n)-diagrams and multiplication law $\alpha\beta = \delta^{n(\alpha,\beta)} \alpha \circ \beta$. We could say it is the twisted monoid group algebra for the monoid $(D(n, n), \circ, 1)$ and the cocycle δ^n. We have thus at our disposition two other series of abstract algebras with parameter, subalgebras of the Brauer algebra:

$P(n, \delta) =$ The subalgebra spanned by planar diagrams also called the Temperley-Lieb algebra $TL(n, \delta)$, in fact invented as diagrams by Kauffmann ([K]).

$A(n, \delta) =$ The subalgebra spanned by annular diagrams.

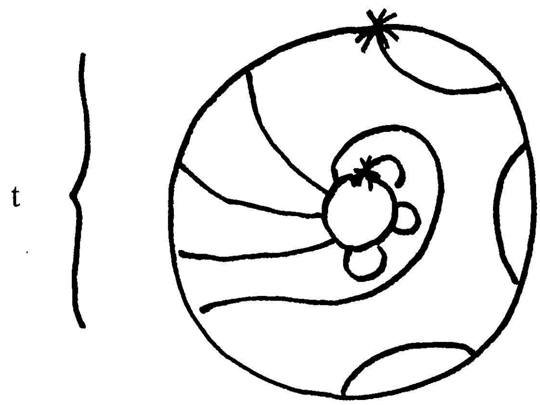
The structure of the Brauer algebra has been studied extensively. See [W], [HW] for much information, and $P(n, \delta)$ is particularly well understood (see [GW], [GHJ]). In this section we will give the structure of $A(n, \delta)$ whenever it is semisimple (over $\mathbf{C}$). It will be worthwhile to call the algebra simply $A(\mathrm{n})$ in this section since we will only consider a fixed δ ($\neq 0$).

Definition 2.1. (i) We call $E(n, t)$ the diagram (in $\mathscr{A}(n, n; t)$)



(so that $E(n, n) = 1$).

(ii) We call $V(n, t)$ the diagram (in $\mathscr{A}(n, n; t)$)



(so that $u = V(n, n)$ and $E(n, 0) = V(n, 0)$).

Note: the role of $*$ is unimportant, it serves only to have a well defined element.

LEMMA 2.2. *Let* $e_t \in A(n)$ *be* $\delta^{-\left(\frac{n-t}{2}\right)} E(n,t)$ *and* $v_t \in A(n)$ *be* $\delta^{-\left(\frac{n-t}{2}\right)} V(n,t)$. *Then*

(i) $e_t^2 = e_t$.

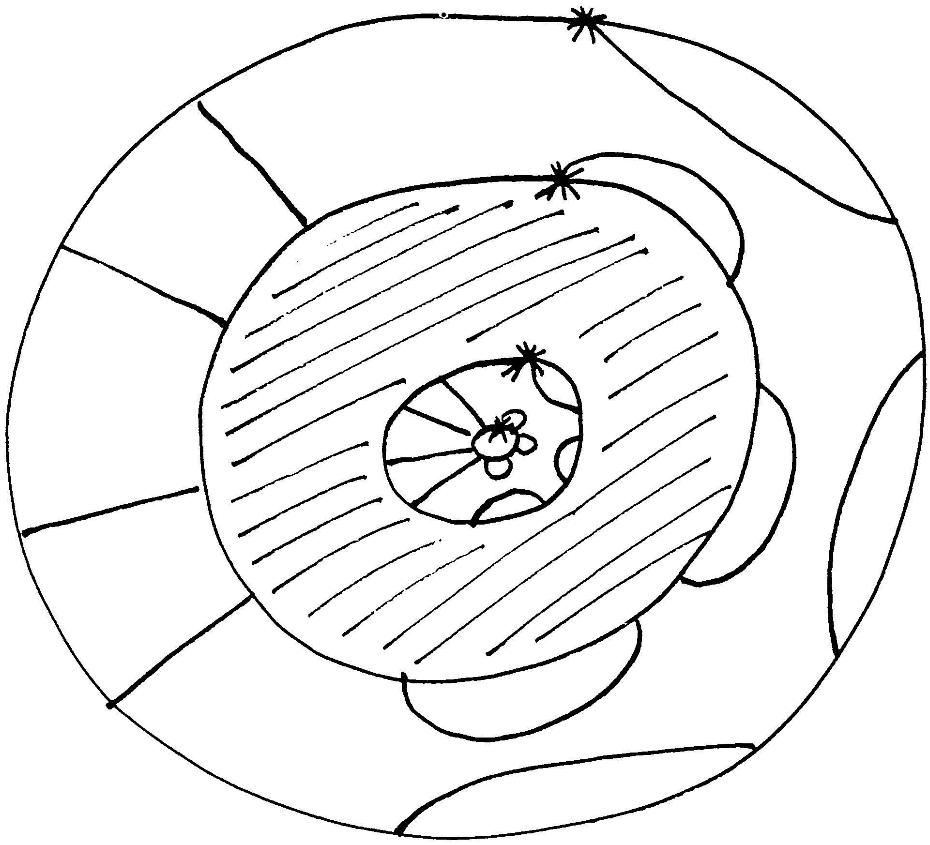
(ii) $(v_t)^t = e_t$ *(so* $e_t v_t = v_t e_t$*).*

(iii) $E(n,t) \circ \mathscr{A}(n,n) \circ E(n,t) \subset \cup_{j<t} \mathscr{A}(n,n;j)$
$\cup \{V(n,n)^k \mid k = 0, 1, 2, \dots, t-1\}$.

(iv) *If* $D \in \mathscr{A}(n,n;t)$, *there are* D_1 *and* D_2 *in* $\mathscr{A}(n,n,t)$ *with* $D = D_1 \circ E(n,t) \circ D_2$.

Proof. (i) and (ii) are evident from diagrams and the multiplication structure in $A(n)$.

(iii) For any D in $\mathscr{A}(n,n)$, $x = E(n,t) \circ D \circ E(n,t)$ is as below.



where there is any annular diagram in the intermediate annulus (shaded). But we see that if x has t through-strings, the intermediate system must connect all of the outer through-strings to one of the inner ones. Once one connection is fixed, all the others must follow in cyclic order, so x is a power of V (with respect to $\circ$).

(iv) As in the proof of Corollary 1.16, we may write $D = E_1 \circ E_2$ with $E_1 \in \mathscr{A}(n,t;t)$, $E_2 \in \mathscr{A}(t,n;t)$. But then pulling the strings around in the middle and introducing $\frac{n-t}{2}$ isolated circles we see that D admits the desired decomposition. $\square$

We proceed to determine the structure of $A(n, \delta)$ when it is semisimple. Note first that the through-strings give a filtration of $A(n)$ by ideals.

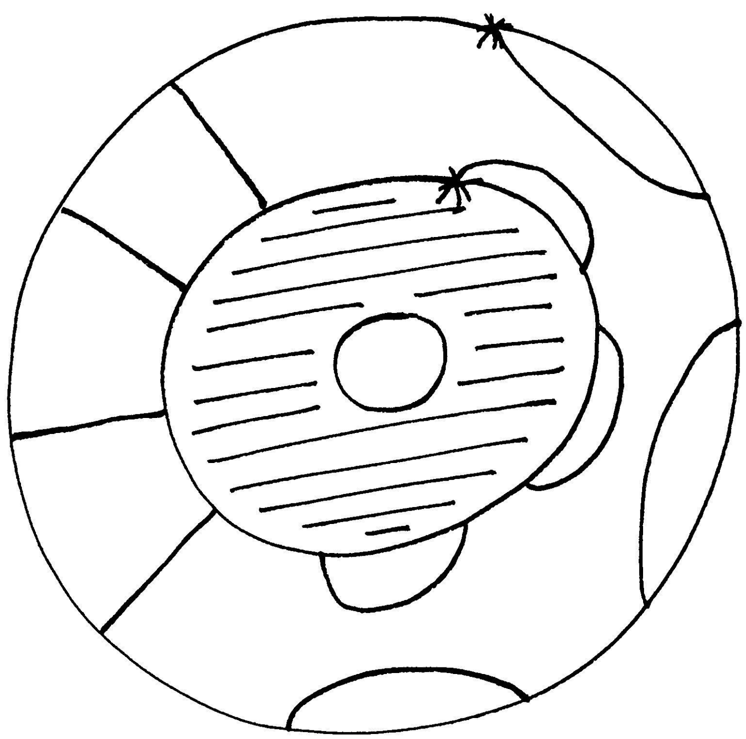
Definition 2.3. $A(n; t)$ is the two-sided ideal linearly spanned by diagrams with $\leqslant t$ through-strings.

Thus if $A(n)$ is semisimple, it is isomorphic to the direct sum $\bigoplus_{t=0}^{n} \frac{A(n;t)}{A(n;t-2)}$, and to determine its structure it suffices to determine that of the quotients, which of course are all semisimple.

THEOREM 2.4. *If* δ *is such that* $A(n, \delta)$ *is semisimple,*

$$\frac{A(n,t)}{A(n,t-2)} \cong \begin{cases} \text{A matrix algebra of size } \operatorname{cat}\binom{n}{\frac{n}{2}} \text{ if } t = 0 \text{ and } n \text{ even.} \\ \text{The sum of } t \text{ matrix algebras of size } \binom{n}{\frac{n-t}{2}} \text{ if } t > 0 \\ \text{(and } n-t \text{ even).} \end{cases}$$

Proof. Suppose first $t > 0$. Let A stand for $A(n,t)/A(n,t-2)$ for short and let it be isomorphic to $\bigoplus_{i=1}^{r} M_{d_i}(\mathbf{C})$. Identify elements of $A(n,t)$ with their classes modulo $A(n,t-2)$. Then by (iv) of Lemma 2.2, the 2-sided ideal generated by e_t is all of $\bigoplus_{i=1}^{r} M_{d_i}(\mathbf{C})$ so we can write $e_t = \bigoplus_{i=1}^{r} p_i$ with p_i a non-zero idempotent in each $M_{d_i}(\mathbf{C})$. But A is linearly spanned by the diagrams in $\mathscr{A}(n,n;t)$ so by (ii) and (iii) of 2.2, $e_t A e_t$ is abelian of dimension t. *Thus each of the* p_i's is a minimal idempotent, $r = t$ and of course $\sum_{i=1}^{t} d_i^2 = t \binom{n}{\frac{n-t}{2}}^2$ by (1.16). But also $\mathscr{A}(n,n;t) \circ E(n,t)$ is exactly all diagrams of the form



so that the ones representing non-zero elements of A are in bijection with $\mathscr{A}(n, t; t)$. Hence $\dim(Ae_t) = |\mathscr{A}(n, t; t)| = t\binom{n}{\frac{n-t}{2}}$. However, $(\oplus_{i=1}^{t} M_{d_i}(\mathbf{C}))\ (\oplus_{i=1}^{t} p_i)$ is a vector space of dimension $\sum_{i=1}^{t} d_i$, so we have

$$\sum_{i=1}^{t} d_i^2 = t\binom{n}{\frac{n-t}{2}} \quad \text{and} \quad \sum_{i=1}^{t} d_i = t\binom{n}{\frac{n-t}{2}}.$$

Thus each of the d_i's is equal to $\binom{n}{\frac{n-t}{2}}$ (e.g. by the "equality" case of the Cauchy Schwartz inequality $(\Sigma d_i \cdot 1) \leqslant \sqrt{\Sigma d_i^2}\sqrt{t}$). This proves the theorem for $t > 0$. The case $t = 0$ follows from the same argument, using $\dim(\mathscr{A}(n, n; 0)) = \operatorname{cat}(n)^2$ and $\dim(\mathscr{A}(n, n; 0)e_0) = \operatorname{cat}(n)$. $\square$

Note that one could avoid the slightly clumsy Cauchy-Schwartz argument by showing that the commutant of $\mathbf{C}[\mathbf{Z}/t\mathbf{Z}]$ is $A(n)$, which is not hard.

Remark 2.5. In fact it is clear from the proof that the algebra $e_t(A(n, t)/A(n, t-1))e_t$ is naturally isomorphic to the group algebra $\mathbf{C}[\mathbf{Z}/t\mathbf{Z}]$, so that the various matrix algebras in $A(n, t)/A(n, t-2)$ are naturally indexed by the t-th roots of unity.

Remark 2.6. In view of 2.5, another way of stating Theorem 2.4 is to say that, if $A(n, \delta)$ is semisimple, its irreducible representations are parametrised by

(i) the number of through-strings t

(ii) a t-th root of unity ω.

Moreover the irreducible representation $\pi = \pi_{t,\omega}$ corresponding to (t, ω) is characterised by the fact that $\pi(v_t) = \omega\pi(e_t)$, and may be given quite explicitly as follows:

If W is the vector space spanned by $\mathscr{A}(t, n; t)$, W becomes an $A(n) - \mathbf{C}[\mathbf{Z}/t\mathbf{Z}]$ bimodule under the left and right action:

$$D \cdot E \cdot F = \begin{cases} D \circ E \circ F & \text{for } D \in \mathscr{A}(n, n) \text{ and } F \in \mathscr{A}(t, t; t), \text{ identified} \\ & \quad \text{with } \mathbf{Z}/t\mathbf{Z}. \\ 0 & \text{if } D \circ E \text{ has } < t \text{ through-strings.} \end{cases}$$

Then if $P_\omega = \frac{1}{t}\sum_{i=1}^{t} \omega^{-i} u^i$ (u as in 1.10), $\pi_{t,\omega}$ is left multiplication on VP_ω.

We give the structure of the subalgebra $\overrightarrow{A(n)}$ of $A(n)$ spanned by oriented diagrams. With obvious notation the result is

THEOREM 2.7. *If* δ *is such that* $\overrightarrow{A(n,\delta)}$ *is semisimple, (n even),*

$$\overrightarrow{A(n,t)}/\overrightarrow{A(n,t-2)} \cong \begin{cases} \text{A matrix algebra of size } \operatorname{cat}\binom{n}{2} & \text{if } t = 0. \\ \text{The sum of } \frac{t}{2} \text{ copies of a matrix algebra} & \\ \text{of size } \binom{n}{\frac{n-t}{2}} \text{ if } t > 0. & \end{cases}$$

Proof. One can simply repeat the proof of Theorem 2.4, the only difference being that the role of the element v would be played by v^2. One could also deduce 2.7 from 2.4 in several ways. One is to note that $\overrightarrow{\mathscr{A}(n)}$ is the fixed point algebra for an involutive automorphism of $\mathscr{A}(n)$ sending u to $-u$. Another way is to observe that the irreducible representations of $\mathscr{A}(n)$ parametrised by (t,ω) $(t>0)$ remain inequivalent for $\omega = exp\left(\frac{2\pi\sqrt{-1}j}{n}\right)$, $j = 0, 1, \dots \frac{n}{2} - 1$ on restriction to $\overrightarrow{\mathscr{A}(n)}$. This is because $v_t^2 = \omega^2 e_t$ in that representation. Then adding the sums of squares of the dimensions one gets the number of oriented diagrams by 1.20. $\square$

Finally we make some remarks about generators and relations. As we saw in the introduction, if we put $f_i = u^i e_{n-2} u^{-i}$ (and $F_i = u^r E(n-2; n-2) u^{-i}$) for $i = 1, 2, \dots, n$, the f_i's satisfy $f_i^2 = f_i$, $f_i f_{i\pm 1} f_i = \delta^{-\frac{1}{2}} f_i$ so that if $g_i = q f_i - (1 - f_i)$ (for $q + q^{-1} + 2 = \delta^2$), the map $T_i \mapsto g_i$, $\rho \mapsto u$ gives a homomorphism from the affine Hecke algebra of type A_n with parameter q onto the diagram algebra $A(n, 2 + q + q^{-1})$. Thus in particular we have constructed some very explicit irreducible representations of the affine Hecke algebra, for certain values of q.

One reason, besides subfactors, for looking at oriented diagrams in the even case is that they allow us to determine the subalgebra generated by $f_1, f_2, \dots, f_n$ (or $g_1, \dots, g_n$).

LEMMA 2.8. *If* n *is even the following three algebras are equal (even if* $A(n,\delta)$ *is not semisimple).*

(i) *The subalgebra of* $A(n)$ *generated by* $f_1, f_2, \dots, f_n$.

(ii) *The two-sided ideal generated by* f_1 *in* $A(n)$.

(iii) $\overrightarrow{A(n, n-2)}$.

Proof. The equality of (ii) and (iii) follows from a special (oriented) case of (iv) of 2.2.

The algebra of (iii) contains the f_i's by definition. That (iii) implies (i) will follow if we can show that any element of $\cup_{t<n} \overrightarrow{\mathscr{A}(n, n; t)}$ is expressible as a product of F_i's. That this is true for diagrams having a straight through-string is a well known fact about the Temperley-Lieb algebra. But if D is an oriented diagram with less than n through-strings, either D has zero through-string and we are in the Temperley-Lieb situation, or $D \circ u^k$ has a straight through-string for some even k. Thus Du^k is a word on the F_i's and it suffices to show that $F_i u^2$ is a word on the F_i's for all i. It follows from a picture that $F_i u^{-2} = F_i F_{i+1} \ldots F_n F_1 F_2 \ \ldots F_{i-2}$. $\square$

Remark 2.9. We leave it to the reader to show that Lemma 2.8 is true without the $\rightarrow$'s if n is odd.

Remark 2.10. It follows from 2.8 that the elements υ_t are in the algebra generated by the F_i's for $t < n$. We record the expression

$$\upsilon_{n-2}^2 = F_n \circ F_1 \circ F_2 \circ \cdots \circ F_n .$$

Thus rotations are unavoidable even if one is only interested in the structure of the algebra generated by the F_i's.

3. The Brauer representation

So far we have begged the important question of when the algebra $A(n, \delta)$ is semisimple. We do not have a complete answer for this but we shall show that it is semisimple whenever δ is an integer $\geqslant 3$, (and that $A(n, -2)$ is not semisimple for $n \geqslant 3$) by using a representation onto a C^*-algebra which we will show to be faithful for such δ. That the representation is faithful for n fixed and large integral (hence any large) δ is rather easy.

Definition 3.1. Let V be a vector space of dimension k and basis $w_1, w_2, \ldots, w_k$. If the diagram $D \in D(n, n)$ has n connecting edges called ε, define $\beta(D) \in \mathrm{End}(\otimes^n V)$ by the matrix (with respect to the basis $\{w_{a_1} \otimes w_{a_2} \otimes \cdots \otimes w_{a_n} \mid a_i = 1, 2, \ldots k\}$ of $\otimes^n V$)

$$\beta(D)_{a_1 a_2 \ldots a_n}^{a_{n+1} \ldots a_{2n}} = \prod_{\varepsilon} \delta(a_{s(\varepsilon)}, a_{b(\varepsilon)})$$

where $s(\varepsilon)$, $b(\varepsilon)$ are the two ends of the edge ε, labelled from 1 to $2n$, and, just in this formula, δ is the Kronecker δ.

LEMMA 3.2. *$D \mapsto \beta(D)$ defines a homomorphism of $B(n, k)$ (hence $A(n, k)$) onto a C^*-subalgebra of* $\mathrm{End}(\otimes^n V)$.