2. Proof of the Theorem

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hence so is $g^{-1}Hg\Gamma$. This shows that $Hz = Hg\Gamma/\Gamma$ is closed, thus proving the proposition.

We recall that if φ is a homeomorphism of a topological space X then $x \in X$ is said to be a *recurrent* point for φ if there exists a sequence $\{n_k\}$ of natural numbers such that $n_k \to \infty$ and $n_k x \to x$. For the proof of the theorem we also need the following general fact.

1.7. PROPOSITION. *Let φ be a homeomorphism of a compact metric space X. Then there exists a recurrent point for φ.*

Proof. Given X and φ as in the hypothesis there exists a φ-invariant probability (Borel) measure on X (cf. [DGS], Proposition 3.8, for instance). The Proposition now follows from the Poincare recurrence theorem; see [M], Theorem 2.3, for a version of the Theorem in the form required here.

It can be seen, by perusing the proofs of the results quoted, from the references mentioned, that the above proof is indeed independent of the axiom of choice. For expositional purposes we also give in the Appendix a more self-contained proof of the Proposition. For this we use the same general idea as above but argue with invariant integrals (positive linear functionals on the space of continuous functions) constructed from the data, without actually using any measure theory.

Incidentally, it may be noted that the assertion in the Proposition is obvious if we assume Zorn's lemma, since in that case there exist compact minimal (nonempty) φ-invariant subsets of X and any point of such a subset is a recurrent point.

2. PROOF OF THE THEOREM

We will prove the Theorem after some technical preparation.

2.1. PROPOSITION. *Let $x \in G/\Gamma$ and $X = \overline{Hx}$. Let $y \in X$ and suppose that there exists a neighbourhood Ω of I in G such that $\{g \in \Omega \mid gy \in X\} \subseteq HV_2$. Then at least one of the following conditions holds:* (i) *Hx is open in X and $y \in Hx$,* (ii) *Hy and $DV_1 y$ are open in their (respective) closures or* (iii) *$Vy \subseteq X$.*

Proof. First suppose that in fact there exists a neighbourhood Ω' of I in G such that $\{g \in \Omega' \mid gy \in X\} \subseteq H$. Then Hy is open in X. Since Hx is dense in X it follows that $Hx = Hy$. Then Hx is open in X and $y \in Hx$, so condition (i) holds in this case. We may therefore assume that there does not exist any neighbourhood Ω' as above. In view of Proposition 1.5, (i), and the H-invariance of X this implies that there exists a sequence $\{v_i\}$ in $V_2 - \{I\}$ such that $v_i \to I$ and $v_i y \in X$ for all i.

Observe that if $V_1(V \cap \Gamma_y)$ is dense in V, then clearly $Vy \subseteq \overline{V_1 y} \subseteq X$ so condition (iii) is satisfied. We may therefore assume that it is not the case. Hence by Lemma 1.4, (ii), there exists a neighbourhood Θ of I in V_2 such that $B\Theta \cap \Gamma_y \subseteq B$. By replacing Ω as in the hypothesis by a smaller neighbourhood we may assume that Ω is open and $\Omega \cap HV_2 \subseteq (\Omega \cap H)\Theta$, the latter being possible because of Proposition 1.5, (i). Now let $g \in H$ be any element such that $gy \in \Omega y$; then there exist $h \in \Omega \cap H$ and $v \in \Theta$ such that $hv \in \Omega$ and $gy = hvy$. Hence $gv_i y = (gv_i g^{-1})gy = (gv_i g^{-1})hvy$. Since $gv_i g^{-1} \to I$ and Ω is a neighbourhood of hv it follows that $gv_i g^{-1}hv \in \Omega$ for all large i. Also $gv_i g^{-1}hvy = gv_i y \in X$ and hence by the hypothesis we get that for all large i, $gv_i g^{-1}hv \in HV_2$ and hence $v_i g^{-1}h \in HV_2$. Since $v_i \neq I$, for any i, by Proposition 1.5, (ii), this implies that $g^{-1}h \in B$. Then $g^{-1}hv \in B\Theta$. Also, since $gy = hvy$, $g^{-1}hv \in \Gamma_y$. By the choice of Θ these two conditions imply that $v = I$. Hence $gy = hy$. This shows that $Hy \cap \Omega y \subseteq (\Omega \cap H)y$. Similarly, since we had $g^{-1}h \in B$, it also shows that $By \cap \Omega y \subseteq (\Omega \cap B)y$. These conditions imply that Hy and By are open in their closures and since DV_1 is open in B it also follows that $DV_1 y$ is open in its closure; therefore condition (ii) holds in this case. This proves the Proposition.

2.2. PROPOSITION. *Let $x \in G/\Gamma$ be such that Hx is not closed and let $X = \overline{Hx}$. Let Y be a compact V_1-invariant subset of X and let $y \in Y$ be recurrent for the action of some $u \in V_1 - \{I\}$. Suppose that either Hx is not open in X or $y \notin Hx$. Then either $Vy \subseteq X$ or $I \in \overline{\{g \in G - HV_2 \mid gy \in X\}}$.*

Proof. Suppose that the assertion does not hold. Then Vy is not contained in X and there exists a neighbourhood Ω of I in G such that $\{g \in \Omega \mid gy \in X\} \subseteq HV_2$. Then Proposition 2.1 and the condition as in the hypothesis imply that Hy and $DV_1 y$ are open in their respective closures and $y \notin Hx$. Let $\Phi = DV_1 \cap \Gamma_y$. Since $DV_1 y$ is open in its closure, it is locally compact and hence $g\Phi \mapsto gy$ is a homeomorphism of DV_1/Φ on to $DV_1 y$ commuting with DV_1-action on the two spaces (cf. [MZ], Section 2.13, for instance). By hypothesis there exists a $u \in V_1 - \{I\}$ such that y is recurrent for the action of u. The preceding observation therefore implies that Φ is recurrent for the action of u on DV_1/Φ. It is easy to see that this can not happen if Φ is contained in vDv^{-1} for some $v \in V$. Applying Lemma 1.4, (i), we can conclude therefore that Φ is a nontrivial subgroup contained in V_1. Therefore y is a V_1-periodic point. Since $y \in X = \overline{Hx}$ and $\{g \in \Omega \mid gy \in X\} \subseteq HV_2$ it follows that there exists a sequence $\{v_i\}$ in V_2 such that $v_i \to I$ and $v_i y \in Hx$ for all i. For each i we have $v_i y \in Hx = Hv_1 y$ and therefore there exists a sequence $\{h_i\}$ in H such that $v_i y = h_i v_1 y$ for all i. Let $i \geqslant 1$ be arbitrary. Let $\Delta_i = H \cap \Gamma_{v_i y}$. Clearly Δ_i contains Φ and by the above relation it also contains $h_i \Phi h_i^{-1}$. Since $v_i y$ is V_1-periodic and $Hv_i y = Hx$ is not closed, by Proposition 1.6 Δ_i must be contained in B. Since $h_i \Phi h_i^{-1}$ is contained in Δ_i and consists of unipotent elements, this implies that $h_i \Phi h_i^{-1} \subseteq V_1$. This implies that $h_i \in B$ (since the subspaces spanned by $\{e_1\}$ and $\{e_1, e_2\}$ have to be h_i-invariant). Therefore there exist $d_i \in D^*$ and $u_i \in V_1$ such that $h_i = d_i u_i$. Now, since $h_i \Phi h_i^{-1} = \Gamma_{h_i v_1 y} \cap V_1 = \Gamma_{v_i y} \cap V_1 = \Phi$ and since Φ is a nontrivial subgroup of V_1 it follows that the diagonal entries of d_i are ± 1. Since $v_1 y$ is a V_1-periodic point, the preceding conclusion implies that the sequence $\{h_i v_1 y\}$ has a limit point in $Hv_1 y = Hx$. But $h_i v_1 y = v_i y \to y$ and therefore we get that $y \in Hx$, contradicting an earlier conclusion. This shows that the Proposition must hold.

Proof of the Theorem. We shall asssume that Hx is not closed and that X does not contain any V-orbit, since in either of these cases there is nothing more to be proved. Let $X' = X - Hx$ if Hx is open in X and $X' = X$ otherwise. Then X' is a closed nonempty V_1-invariant subset of X. By Propositions 2.2 and 1.7 any compact V_1-invariant subset of X' contains a y such that $I \in \overline{\{g \in G - HV_2 \mid gy \in X\}}$. Let $\{r_i\}$ be an enumeration of the set of all rational numbers. We now construct a decreasing sequence $\{Y_k\}$ of compact V_1-invariant subsets of X' and a sequence $\{t_k\}$ of rational numbers as follows. Recall that by Proposition 1.3 X' contains a compact nonempty V_1-invariant subset. Let Y_1 be such a subset and let $t_1 = 0$. After the sets $Y_1, \ldots, Y_k$ and the numbers $t_1, \ldots, t_k$ are chosen, for some $k \geqslant 1$, we proceed to choose Y_{k+1} and t_{k+1} as follows. As observed above, Y_k contains a point y such that $I \in \overline{M}$ where $M = \{g \in G - HV_2 \mid gy \in X\}$. Then by Proposition 1.2 $\overline{HMV_1}$ contains either V_2^+ or V_2^-. Now let i be the smallest natural number satisfying the following conditions: a) $r_i \neq t_j$ for any $j = 1, \ldots, k$ and b) r_i is positive if $V_2^+ \subseteq \overline{HMV_1}$ and negative otherwise. Put $t_{k+1} = r_i$. Then $v_2(t_{k+1}) \in \overline{HMV_1}$ and hence by Lemma 1.1 there exists a $y' \in \overline{V_1 y} \subseteq Y_k$ such that $v_2(t_{k+1})y' \subseteq X$. Put $Y_{k+1} = \overline{V_1 y'}$. This completes the inductive construction of the sequences $\{Y_k\}$ and $\{t_k\}$. It is clear from the construction that $\{Y_k\}$ is a decreasing sequence of compact

V_1-invariant subsets of X and $v_2(t_k)Y_k \subseteq X$ for all k. Also it is easy to see that $\{t_k \mid k \geqslant 1\}$ contains either all positive rational or all negative rational numbers. Now let $Y' = \cap_{k=1}^{\infty} Y_k$. Since $\{Y_k\}$ is a decreasing sequence of compact subsets, Y' is nonempty. Now if $\{t_k \mid k \geqslant 1\}$ contains all positive rational numbers then $v_2(r)Y' \subseteq X$ for all positive rational numbers r and hence by continuity $V_2^+ Y' \subseteq X$ and, similarly, in the alternative case $V_2^- Y' \subseteq X$. This completes the proof of the theorem.

Appendix: Recurrent points

For a compact metric space X we denote by $C(X)$ the space of all continuous real-valued functions on X equipped with the sup-norm topology and by $C(X)^+$ the subset of $C(X)$ consisting of all nonnegative functions; the supremum norm of $f \in C(X)$, namely $\sup\{\,|f(x)|\;|\,x \in X\}$, will be denoted by $\|f\|$. By an integral on $C(X)$ we mean a linear functional on $C(X)$ which takes nonnegative values on $C(X)^+$. For an integral Λ on $C(X)$ the *support* of Λ is defined to be the subset of X consisting of all $x \in X$ such that $\Lambda(f) > 0$ for any $f \in C(X)^+$ for which $f(x) > 0$; the support is easily seen to be a closed subset of X. It can also be verified by a simple point-set topological argument that if Λ is an integral on $C(X)$ and $f \in C(X)$ vanishes on the support of Λ then $\Lambda(f) = 0$. If Λ is an integral on $C(X)$, where X is a compact metrizable space, and X' is the support of Λ then there exists a unique integral Λ' on $C(X')$ such that $\Lambda'(f\,|_{X'}) = \Lambda(f)$ for all $f \in C(X)$, where $f\,|_{X'}$ denotes the restriction of f to X'; this follows from the Tietze-Urysohn extension theorem (cf. [D], (4.5.1)) and the above mentioned property of the support. We note also that the support of Λ' as above is the whole of X'.

For any homeomorphism φ of a compact (metrizable) space X an integral Λ on $C(X)$ is said to be φ-invariant if $\Lambda(f \circ \varphi) = \Lambda(f)$ for all $f \in C(X)$; clearly the support of a φ-invariant integral on $C(X)$ is a φ-invariant (closed) subset of X.

Proof of Proposition 1.7. We fix a dense sequence in $C(X)$, say f_j, $j = 1, 2, \ldots$. Let $x_0 \in X$. Given f_j, for any sequence $\{m_k\}$ of natural numbers $m_k^{-1} \sum_{i=0}^{m_k - 1} f_j \circ \varphi^i(x_0)$ is a bounded sequence and therefore admits a convergent subsequence. Using a standard procedure (finding $\{m_k^{(j)}\}$, with each sequence a subsequence of the previous one, such that the corresponding sequence for f_j as above converges and considering $\{m_k^{(k)}\}$) we get a sequence $\{n_k\}$ of natural numbers such that $n_k^{-1} \sum_{i=0}^{n_k - 1} f_j \circ \varphi^i(x_0)$ converges for all j; also, the limit is between $-\|f_j\|$ and $\|f_j\|$. Since $\{f_j\}$ is dense