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THE THEOREM OF KERÉKJÁRTÓ ON PERIODIC HOMEOMORPHISMS OF THE DISC AND THE SPHERE

by Adrian CONSTANTIN and Boris KOLEV

ABSTRACT. We give a modern exposition and an elementary proof of the topological equivalence between periodic homeomorphisms of the disc and the sphere and euclidean isometries.

1. INTRODUCTION

In 1919, Kerékjártó published the first proof of the topological equivalence between periodic homeomorphisms of the disc and the sphere and euclidean isometries [3]. In the same journal just following Kerékjártó's article, Brouwer [1] gave his own argument for these theorems, explaining that these results had been known to him for a long time and that they were consequences of some earlier and slightly different theorems of his on periodic homeomorphisms of compact surfaces. However, Brouwer's proof is not easy to follow and the proof of Kerékjártó was just sketched and contained a gap.

It was only in 1934 that a complete proof of this important theorem was presented by Eilenberg [6]. More recently Epstein [7] has reconsidered the question for pointwise periodic homeomorphisms (each point is periodic under f but the period $n(x)$ depends on x and may not be bounded). Because of the importance of these results and since no modern exposition of them seems to be found in the litterature, the authors have thought that it would be useful to present a modern and elementary proof. The essential arguments, however, remain those of [1, 3, 6].

2. Background and Definitions

Let X be a topological space and f a homeomorphism of X. We say that f is periodic if there is an integer $n > 0$ such that $f^n = Id$. The period of f is the smallest positive integer n with this property.

As we will use them without further justifications, let us first recall some basic properties of one-dimensional maps.

Let $f : I \to I$ be a periodic homeomorphism of the unit interval. If f preserves the endpoints then f is the identity map. If f exchanges the endpoints then $f^2 = Id$ and f is conjugate to the reflection map $x \mapsto 1 - x$. Similarly, a periodic homeomorphism of the real line $\mathbf{R}$ is the identity map or is a conjugate of the involution $x \mapsto -x$ according to whether it is an increasing or a decreasing function.

Let $f : S^1 \to S^1$ be a periodic homeomorphism of period n of the unit circle. If f is order-preserving then the rotation number of f, $\rho(f) = k/n$, where k and n are coprime (see [5] for an excellent exposition on rotation numbers) and f is conjugate to a rotation of angle $2k\pi/n$. If f is order-reversing then f has exactly two fixed points, f^2 is the identity map and the two arcs delimited on S^1 by the fixed points of f are permuted by f.

A metric space X is path connected if there exists a continuous map from the unit interval [0, 1] into X which joins any two given points. It is arcwise connected if there is a topological embedding of [0, 1] into X which joins any two given distinct points. In fact, it can be shown that the two notions are equivalent (see [14, Theorem 4.1] or [11, Lemma 16.3]).

LEMMA 2.1. *A metric space X is path connected if and only if it is arcwise connected.*

A useful characterisation of path connected spaces is given in term of local connectivity. A metric space X is locally connected if each point of X possesses arbitrary small connected neighbourhoods. The following can be shown (see [8, Theorem 3.15] or [11, Lemma 16.4]):

LEMMA 2.2. *A compact, connected and locally connected metric space is pathwise connected.*

Another important ingredient used in this article, and in fact the ultimate result we will need, is the famous Jordan-Schoenflies theorem on simple closed curves in the plane (see [2, 9] or [12, Theorem 17.1]).

THEOREM 2.3 (Jordan-Schoenflies). *Every simple closed curve J divides the plane into exactly two components of each of which it is the* *complete boundary and the closure of the bounded component can be mapped topologically onto the closed unit disc.*

In what follows, a *closed* topological disc (or just a topological disc) D is the image under a topological embedding of the *closed* unit disc and we write D^o for its interior and ∂D for its boundary. However, the closure of a bounded open set which is homeomorphic to the open unit disc is not necessarily a closed topological disc [11, Chapter 15].

PROPOSITION 2.4. *Let* $D_1, D_2, \ldots, D_n$ *be a finite number of closed topological discs in the plane and* J^o *be any connected component of* $\bigcap_{i=1}^n D_i^o$. *Then* ∂J *is a simple closed curve and* J *the closure of* J^o *is a topological disc.*

Proof of 2.4. We will use induction on n, the number of discs. If $n = 1$ this is just the Jordan-Schoenflies theorem, so let us suppose that the result holds for some $n (n \geqslant 1)$ and let J^o be any component of the complement of $n+1$ topological discs $D_1, D_2, \ldots, D_{n+1}$ in the plane. Let K^o be the component of $\bigcap_{i=1}^n D_i^o$ that contains J^o. By induction, its closure K is a topological disc. Since J^o is a component of $K^o \cap D_{n+1}^o$, it suffices to show that the result holds for two discs D_1 and D_2 (see Figure 1). Set $C_i = \partial D_i$ for $i = 1, 2$ and let J be the closure of a component of $D_1^o \cap D_2^o$. We have that $\partial J \neq \emptyset$ and $\partial J \subset C_1 \cup C_2$. If ∂J is entirely contained in one of the two curves, say C_1, then $J = D_1$ and the lemma is proved. We can thus suppose that $\partial J \not\subset C_1$ and $\partial J \not\subset C_2$.

Let $x \in \partial J$, $x \notin C_2$. Then $x \in C_1 \cap D_2^o$, and we can find an arc γ in C_1 such that:

$$x \in \gamma\,, \quad \gamma \subset \partial J\,, \quad \gamma \backslash \partial\gamma \subset D_2^o, \quad \partial\gamma \subset C_2\,.$$

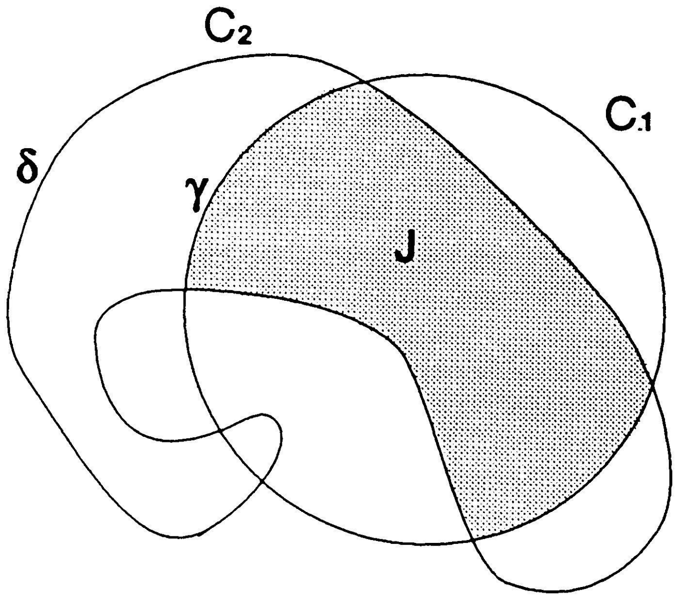

FIGURE 1

The endpoints of γ determine on C_2 an arc δ disjoint from J^o and such that $\delta \cap J = \partial\delta$. We note that there is an at most countable family of such arcs γ, noted $(\gamma_i)_{i \in N}$ and that $diam(\gamma_i) \to 0$ as $i \to \infty$. The boundary of J is the simple closed curve obtained from C_2 when substituting the arcs γ_i for the arcs δ_i and J is a topological disc by the Jordan-Schoenflies theorem. □

The following remarkable property of periodic homeomorphisms which is a direct consequence of 2.4 is true in a more general setting than the plane $\mathbf{R}^2$, namely in topological manifolds of dimension 2 because of its local nature. We will give it in that context since we will use it for the disc and the sphere, repeatedly in this article.

LEMMA 2.5. *Let* $f: S \to S$ *be a periodic homeomorphism of an arbitrary 2-dimensional topological manifold* S *and let* $x \in Fix(f)$, *a fixed point of* f. *Then for any neighbourhood* N *of* x, *there exists a topological disc* Δ_x *such that:*

1. $\Delta_x \subset N$,

2. Δ_x *is a neighbourhood of* x,

3. $f(\Delta_x) = \Delta_x$.

Proof of 2.5. We can first assume that N and its image under f, $f(N)$, are contained in some local chart U homeomorphic with $\mathbf{R}^2$ and will continue to call x and N the corresponding point and set in $\mathbf{R}^2$. Let D_x be an euclidean disc of centre x and radius η where $\eta > 0$ is chosen such that $f^k(D_x) \subset N$ for $k = 0, 1, .., n-1$ and let C_x be its boundary. Let Δ_x be the closure of the component of the invariant set $\bigcap_{k=0}^{n-1} f^k(D_x^o)$ which contains x. By 2.4, Δ_x is a topological disc which is invariant under f (components are sent to components by a homeomorphism) and satisfies the three assertions of the lemma. □

Remark. The boundary γ_x of Δ_x, which is an invariant simple closed curve, is contained in $\bigcup_{k=0}^{n-1} f^k(C_x)$.

3. PERIODIC HOMEOMORPHISMS OF THE DISC

THEOREM 3.1. *Let* $f: D^2 \to D^2$ *be a periodic homeomorphism. Then there exists* $r \in O(2)$ *and a homeomorphism* $h: D^2 \to D^2$ *such that* $f = hrh^{-1}$.

Before attacking the proof of the result above, let us first look at a special case of Theorem 3.1, namely:

PROPOSITION 3.2. *Let* $f : D^2 \to D^2$ *be a periodic homeomorphism such that* $f/_{\partial D^2} = Id$. *Then* $f = Id$.

Proof of 3.2. Let d be an arbitrary diameter of D^2 with endpoints A and B and let Δ be one of the two connected components of $D^2 - d$. The set:

$$E = \bigcap_{i=1}^{n} f^i(\Delta^o)$$

is invariant under f and the closure of each of its components is a topological disc.

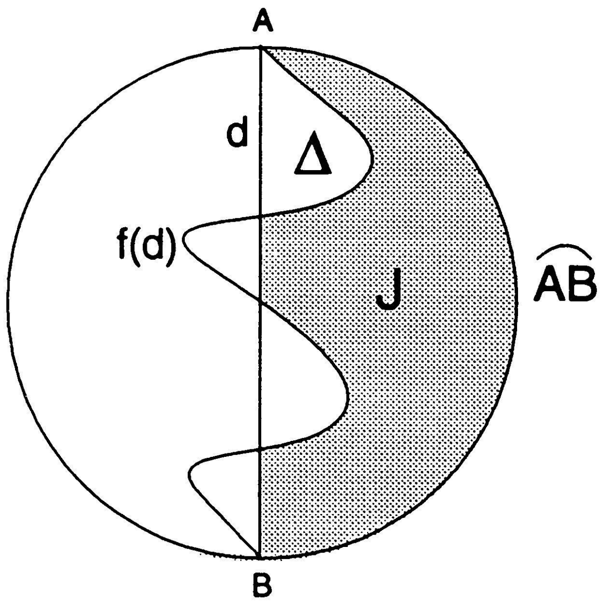

FIGURE 2

Let $\widehat{AB}$ be the arc of circle joining A to B in the boundary of Δ. Since $f^i(\widehat{AB}) = \widehat{AB}$ for all i, there exists a component of E, say J^o, whose closure J contains $\widehat{AB}$ (see Figure 2). By 2.4, J is a topological disc which is invariant under f.

We can write $\partial J = \widehat{AB} \cup \delta$ where δ is an f-invariant, simple arc with endpoints A and B such that:

$$\delta \subset \bigcup_{i=1}^{n} f^i(d) .$$

Since $f(A) = A$ and $f(B) = B$, $f/_\delta = Id$. Let x be a point of the arc δ. There exists $i \in \{1, \ldots, n\}$ such that $x \in f^i(d)$ and $x = f^{n-i}(x) \in d$ so that $\delta = d$ and $f/_d = Id$. Since the diameter d was chosen arbitrarily, we have shown that $f = Id$ on D^2. $\square$

From now on, f will denote a periodic homeomorphism of the disc of period n with $n > 1$. In the sequel of this section, we prove Theorem 3.1, first investigating the structure of the fixed point set of f.

PROPOSITION 3.3. *Suppose* $f : D^2 \mapsto D^2$ *is a periodic homeomorphism of period* n $(n > 1)$; *then:*

1. *if* f *is orientation-preserving,* $Fix(f)$ *is reduced to a single point which is not on the boundary of* D^2 *and for* $1 \leqslant i \leqslant n-1$, $Fix(f^i) = Fix(f)$;
2. *if* f *is orientation-reversing,* $f^2 = Id$ *and* $Fix(f)$ *is a simple arc which divides* D^2 *into two topological discs which are permuted by* f.

Proof of 3.3. Suppose first that f is orientation-preserving. By Brouwer fixed point theorem, f has at least one fixed point. Since $f/_{\partial D^2}$ is orientation-preserving and periodic, f has no fixed point on ∂D^2. Otherwise f would be the the identity map on ∂D^2 and using 3.2, f would be the identity map on the whole disc which is excluded by hypothesis. Therefore, f has at least one fixed point in $D^2 \backslash \partial D^2$ which we can assume to be, up to conjugacy, O, the center of the disc.

Let $A = D^2 \backslash \{O\}$. A is a half open annulus which is invariant under f. Suppose now that an iterate f^i of f has a fixed point $x_0 \in A$. Let $\tilde{x}_0$ be a lift of x_0 to the universal covering space $\tilde{A}$ of A and G be the lift of f^i such that $G(\tilde{x}_0) = \tilde{x}_0$. G^n is a lift of Id which fixes one point, thus $G^n = Id$. In particular, $G/_{\partial \tilde{A}}$ is a periodic and orientation preserving homeomorphism of the line, thus $G = Id$ on $\partial \tilde{A}$. Therefore, $f^i = Id$ on ∂D^2 and, according to 3.2, $f^i = Id$ on the whole disc, so that i is a multiple of n according to the definition of n.

Suppose now that f is orientation-reversing. In that case, f has exactly two fixed points on ∂D^2 which we denote by A and B and f^2 is the identity map on ∂D^2, therefore, by 3.2, $f^2 = Id$ on D^2 .

We assert that $Fix(f)$ is connected. For if not, we can find two nonempty compact sets K_1 and K_2 such that

$$Fix(f) = K_1 \cup K_2, \qquad K_1 \cap K_2 = \emptyset .$$

If $A \in K_1$ and $B \in K_2$, it is then possible to construct a simple arc γ in $D^2 \backslash (K_1 \cup K_2)$ which intersect ∂D^2 only on its endpoints and which separates A from B. Using the same argument as the one used in the proof of 3.2, we can show the existence of an f-invariant simple arc:

$$\delta \subset \bigcup_{i=0}^{n-1} f^i(\gamma) \subset D^2 \backslash Fix(f)$$

which separates A from B. But f must then have a fixed point on δ which gives a contradiction. Therefore we can suppose that one of the two compact sets, say K_1 is contained in $D^2 \backslash \partial D^2$. In that case, it is possible to construct a simple closed curve $c \subset D^2 \backslash \partial D^2$ which does not meet $K_1 \cup K_2$ and such that the topological disc it bounds contains at least one point of K_1. Using similar arguments as those of the proof of 2.5, we can find an f-invariant topological disc in $D^2 \backslash \partial D^2$ whose boundary contains no fixed point. This gives again a contradiction, since any simple closed curve which bounds an invariant disc has exactly two fixed points of f.

The previous arguments applied to an arbitrarily small invariant topological disc around a fixed point given by 2.5 shows that $Fix(f)$ is also locally connected and by 2.2, $Fix(f)$ is therefore pathwise connected. In view of 2.1, there exists a simple arc γ in $Fix(f)$ which joins A and B. This arc divides D^2 into two topological discs Δ_1 and Δ_2 by the Jordan-Schoenflies theorem. $D^2 \backslash \gamma$ is obviously invariant under f and the two arcs on ∂D^2 delimited by A and B are permuted by f, therefore $f(\Delta_1) = \Delta_2, f(\Delta_2) = \Delta_1$ and $Fix(f)$ is reduced to γ. $\square$

Proof of 3.1. Suppose first that f is orientation-preserving. By 3.3, we can suppose that $Fix(f) = \{O\}$, the center of the disc. Since $f/_{\partial D^2}$ is a periodic homeomorphism of period n, the rotation number of $f/_{\partial D^2}$, $\rho(f/_{\partial D^2}) = k/n$, where k and n are coprime. We are going to prove that f is conjugate to a rotation by angle $2k\pi/n$ around the origin. Without loss of generality, we can assume that $k = 1$. Indeed, suppose the result holds if $\rho(f/_{\partial D^2}) = 1/n$. Then, if $k > 1$ we replace f by f^j where $j \in \mathbf{N}$ is such that $jk \equiv 1 (mod\, n)$. Then $\rho(f^j/_{\partial D^2}) = 1/n$, thus f^j is conjugate to a rotation by angle $2\pi/n$ around the origin and since $(f^j)^k = f$, it follows that f is conjugate to a rotation by angle $2k\pi/n$.

Let us consider the quotient space $D^2/_f$ where two points are identified if they belong to the same orbit under f. $D^2/_f$ is endowed with the quotient topology. It is a compact and pathwise connected metric space, the metric being defined by:

$$d(\pi(x), \pi(y)) = \inf_{0 \leqslant h, k \leqslant n-1} \{d(f^k(x), f^h(y))\},$$

where $\pi : D^2 \to D^2/_f$ is the canonical projection.

By 2.1, we can find a simple arc γ from $\pi(O)$ to an arbitrary point on $\pi(\partial D^2)$. Since the group of homeomorphisms generated by f acts freely on D^2 except at O it follows that $\pi : D^2 \to D^2/_f$ is a regular branched covering (see [10] page 49). Therefore, $\pi^{-1}(\gamma)$ is the union of n disjoint simple arcs (with the exception of their common endpoint O) $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$, which divide D^2 into n disjoint sectors, $A_0, A_1, \ldots, A_{n-1}$. The hypothesis $\rho(f/_{\partial D^2}) = 1/n$ implies that $\gamma_i = f^i(\gamma_0)$.

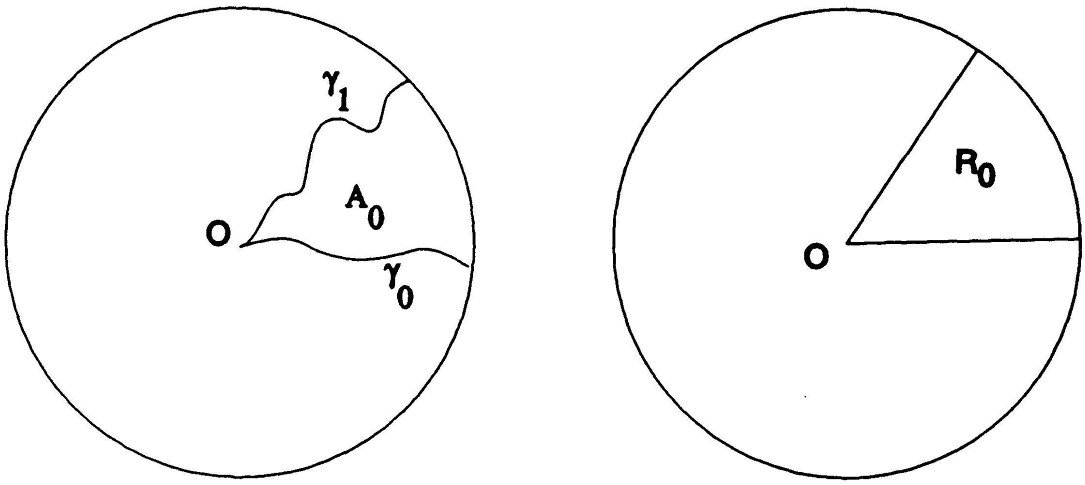

FIGURE 3

Let h be a homeomorphism between A_0 and R_0, the fundamental region in D^2 of the rotation by angle $2\pi/n$ around the origin, and such that $h\backslash_{\gamma_1} = rh\backslash_{\gamma_0}$. We can extend h to a homeomorphism of D^2 by defining $h/_{A_i}$ as $r^i h f^{-i}$, r being the rotation of centre O and angle $2\pi/n$. It is easy to verify that h is an homeomorphism of D^2 and that $f = h^{-1}rh$.

Suppose now that f is orientation-reversing. By 3.3, *Fix*(f) is a simple arc γ which divides D^2 into two topological discs Δ_1 and Δ_2 which are permuted by f. Let h be a homeomorphism between Δ_1 and the upper half disc D_1. We define h on Δ_2 in the following way:

$$h(y) = Sh/_{\Delta_1} f(y), \; y \in \Delta_2 ,$$

where S is the reflection about the x-axis. It is then easy to verify that h is a homeomorphism of D^2 and this gives a conjugacy between f and S. $\square$

Remark. Using 3.1, it can also be shown that any periodic homeomorphism of the annulus is topologically equivalent to an euclidean isometry (modulo a flip of the boundary if it is not boundary-preserving).

4. Periodic homeomorphisms of the sphere

The main result of this section is

Theorem 4.1. *Let* $f : S^2 \to S^2$ *be a periodic homeomorphism. Then there exists* $r \in O(3)$ *and a homeomorphism* $h : S^2 \to S^2$ *such that* $f = hrh^{-1}$.

Proof of 4.1. We will divide the proof of Theorem 4.1 into two cases according to whether or not f has at least one fixed point.

Suppose first that f has a fixed point. Using 2.5, we deduce the existence of an invariant simple closed curve c which divides S^2 into two invariant discs D_1 and D_2.

If f is orientation preserving and $f \neq Id$, then f has no fixed point on c (cf. 3.2). Therefore, by Brouwer's fixed point theorem we know then that f has at least two fixed points; after a conjugacy, we can suppose that f fixes the two poles N and S of S^2. Using the results of last section, we are able to find n arcs joining N and S such that their union is an invariant set under f. As in Section 3, we can then construct a conjugacy between f and a rotation by angle $2k\pi/n$ around the South-North axis.

If f is orientation-reversing, then f has two fixed points on c. In each of the invariant disc D^1 and D^2, the fixed point set of f consists of a simple arc which joins the two fixed points of f on c. The union of these two arcs is a simple closed curve which coincides with the fixed point set of f on S^2. It is then easy to construct a conjugacy between f and the reflection about the equator.

Let now suppose that f has no fixed point on S^2. Up to conjugacy, we can assume that the second iterate of f, f^2 is a periodic rotation around the North-South axis. In particular the points N and S are exchanged by f. For $t \in (-1, 1)$, let C_t be the circle obtained by cutting the sphere by the plane $z = t$, D_t the disc bordered by C_t on S^2 which contains N and:

$$t_0 = \inf\{t \in (-1, 1)\ ;\quad D_t \cap f(D_t) = \emptyset\}\ .$$

We write $D = D_{t_0}$ and $C = C_{t_0}$ for convenience. Then D meets $f(D)$ on its boundary and only on its boundary (see Figure 4). Let $P_0 \in C \cap f(C)$ and $P_1, P_2, \ldots, P_{n-1}$, the orbit of P_0 under f. The points $P_0, P_2, \ldots, P_n$ and $P_1, P_3, \ldots, P_{n-1}$ are distinct because f^2 is a rotation of period $n/2$.

Suppose that there exists $i \in \{1, 3, \ldots, n-1\}$ such that P_0 and $P_i = f^i(P_0)$ coincide. Then P_0, S and N are fixed by f^{2i} so $f^{2i} = Id$. Therefore $2i = n$.

Let b_0 be the arc of great circle that joins N to P_0 in D and $b_{n/2}$ its image under $f^{n/2}$. Then $b = b_0 \cup b_{n/2}$ is a simple arc joining N and S and not meeting its first $(n/2) - 1$ iterates under f away from N and S. These arcs divide the sphere into $n/2$ sectors and we can build a conjugacy between f and the composition of a rotation of period $n/2$ around the North-South axis with a reflexion about the equator.

Suppose now that the points $P_0, P_1, \ldots, P_{n-1}$ are distinct. Let b_0 an arc of great circle joining N and P_0 in D and b_0' an arc joining S to P_0 in $f(D)$ disjoint from $f(b_0)$ and from its first $n - 1$ iterates (which is possible since f^2 is a rotation). The union of these two arcs is again a simple arc joining N and S which does not meet its first $n - 1$ iterates under f away from N and S. The union of this arc and its iterates divides the sphere S^2 into n disjoint sectors. In that case, f is topologically equivalent to the composition of a rotation of period n around the North-South axis with a reflexion about the equator. $\square$

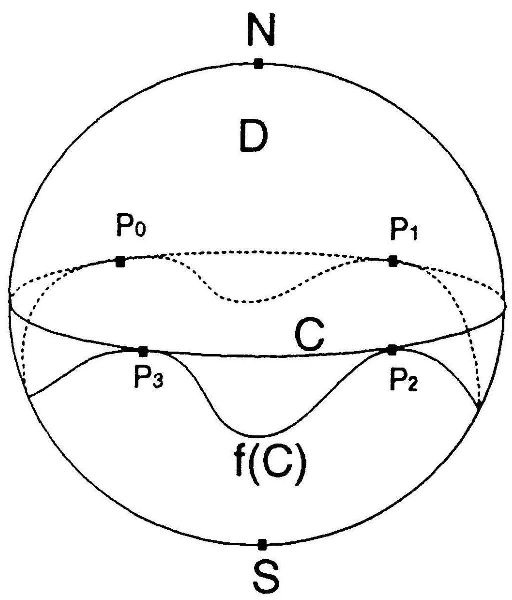

FIGURE 4

COROLLARY 4.2. *Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be a periodic homeomorphism. Then f is topologically conjugate to a finite order rotation around the origin or to the reflexion about the x-axis.*

Proof of 4.2. We can extend f to a homeomorphism of the Sphere S^2 by identifying the plane $\mathbf{R}^2$ with the complement of the North pole using the stereographic projection. Looking at the proof of 4.1, f is either equivalent to a rotation around the North-South pole or to a reflexion about a great circle which we can assume to pass through the north pole N. It is not difficult to show that the conjugacy can be chosen to fix also the North pole N. This equivalence induces, therefore, a topological equivalence between f and a rotation or a reflexion about the x-axis. □

Remark. The investigation of periodic homeomorphisms on surfaces of positive genus has been studied extensively. We cannot give here a complete bibliography on the subject. We would just like to cite original works of Kerékjártó [4] and Nielsen [13] which lead to the conclusion that a periodic homeomorphism of a Riemannian surface of positive genus is conjugate to a conformal isometry.

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