4. PRESENTATIONS I: THE THEORY OF TRANSFORMATION GROUPS

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **40 (1994)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

$$\prod^{r_1'} SL_{ns}(\mathbf{R}) \times \prod^{r_1''} SL_{ns/2}(\mathbf{H}) \times \prod^{r_2} SL_{ns}(\mathbf{C})$$

(where for **H**, SL denotes elements of GL of reduced norm 1). Dividing out the maximal compact subgroups, we find that $S\Gamma$ operates discontinuously on a homogeneous space of dimension

$$r(SA) := r(A) - r(K) ,$$

which may be called the "*reduced geometric unit rank of A*". Explicitly, inferring

$$r(K) = r_1' + r_1'' + r_2 - 1 ,$$

we obtain from (4) the formula

$$(5) \qquad r(SA) = r_1' \frac{(k+2)(k-1)}{2} + r_1'' \frac{(k-2)(k+1)}{2} + r_2(k-1)(k+1) .$$

We will go through the cases of small $r(SA)$ in the concluding section.

It is surprising how easily the existence of a fundamental domain with finitely many neighbors implies another finiteness theorem, which has already been mentioned:

THEOREM 2. Γ *contains only finitely many conjugacy classes of finite subgroups.*

Proof [B1]. Let $G = Gl_n(D_{\mathbf{R}})$ and C be the maximal compact subgroup used above. Let $H < \Gamma$ be a finite subgroup. Then H is contained in a maximal compact $\tilde{C}$, which is conjugate to $C : \tilde{C} = gCg^{-1}$. Then $Cg^{-1}\tilde{C} = Cg^{-1}$, so H fixes the point $P = Cg^{-1}$ of $C \backslash G = H^+$. Let $\gamma \in \Gamma$ be such that $P\gamma \in Z_0$, the fundamental domain. It follows that $P\gamma\gamma^{-1}H\gamma = P\gamma$, so $\gamma^{-1}H\gamma \subset E(Z_0)$, which is finite. (This proof holds for arbitrary arithmetic groups.)

4. PRESENTATIONS I: THE THEORY OF TRANSFORMATION GROUPS

We have already indicated that not only generators but also defining relations can be extracted from a "good" operation of Γ on a "good" space and that reduction theory provides us with both. The basic idea is already inherent in Poincaré's treatment of Fuchsian groups (see e.g. [F], p. 168 ff.). Gerstenhaber [G] established the abstract setting; later contributions are due to Behr [Be 1, Be 2] and Macbeath [Mb]. Abels [A] gave a unified and generalized treatment; the following is taken from there.

Let $T, H, F, E = E(F)$ be as in the beginning of the last section. Let $\tilde{H}$ be the abstract group generated by elements t_h, $h \in E$, with defining relations

$$t_{h_2} \cdot t_{h_1^{-1}} = t_{h_2 h_1^{-1}} = \text{ for all } h_1, h_2 \in E$$
$$\text{such that } Fh_1 \cap Fh_2 \cap F \neq \varnothing .$$

(Be sure that this makes sense!) There is an obvious homomorphism

$$\varphi : \tilde{H} \to H, \ t_h \to h ,$$

which is surjective if the hypotheses of the basic lemma are fulfilled. With a little more luck, φ is an isomorphism.

THEOREM (Gerstenhaber – Behr – Macbeath). *Assume that F and T are connected and T is simply connected. Then φ is an isomorphism if*

(1) $\mathring{F}H = T$ ($\mathring{F}$ = *interior of* F)

or if

(2) $FH = T$, F *is closed and* $\{Fh \mid h \in H\}$ *is a locally finite cover of* T

or if

(3) $\{Fh \mid h \in H\}$ *is an H-denumerable cover of* T.

(For the definition of "H-denumerable cover" and some examples, see [A].) In particular, if one of the three cases is given and E is finite, then H is finitely presented.

The idea of the proof is the following. On the space

$$Z = \{(t, h) \in T \times \tilde{H} \mid th^{-1} \in F\}$$

($\tilde{H}$ operating on T via φ) define an equivalence relation by

$$(t_1, h_1) \sim (t_2, h_2) \Leftrightarrow t_1 = t_2 \quad \text{and} \quad \varphi(h_2 h_1^{-1}) \in E .$$

Then $Y = Z/\sim$ turns out to be a covering space of T with the properties

(i) $\tilde{H}$ acts on Y, and p: class of $(t, h) \to t$ is a $\tilde{H}$-map;

(ii) ker φ acts as a group of decktransformations of p; this action is free and transitive on the fibres of p;

(iii) There is a section for p over F.

This easily implies that φ is injective if Y is the trivial covering of T, and this will be so if T is simply connected. If T is not simply connected, one can say, under suitable hypotheses, something about the kernel of φ; Swan [Sw] constructs an exact sequence

$$1 \to N \to \pi_1(T) \to \tilde{H} \to H \to 1$$

where the kernel N can be described.

It is case (2) of the theorem which directly applies to our unit groups Γ. The space of positive forms on which we made Γ operate is an open convex cone in a Euclidean space and, as such, clearly connected and simply connected. It was pointed out that the fundamental domain could be chosen as the connected union of closed cells. The "third theorem of finiteness" ensures that $F\Gamma$ is a locally finite cover. Thus, we finally have

THEOREM 3. *The group of units of any* $\mathbf{Z}$*-order is finitely presented.*

As mentioned before, this generalizes to arithmetic groups [BHC].

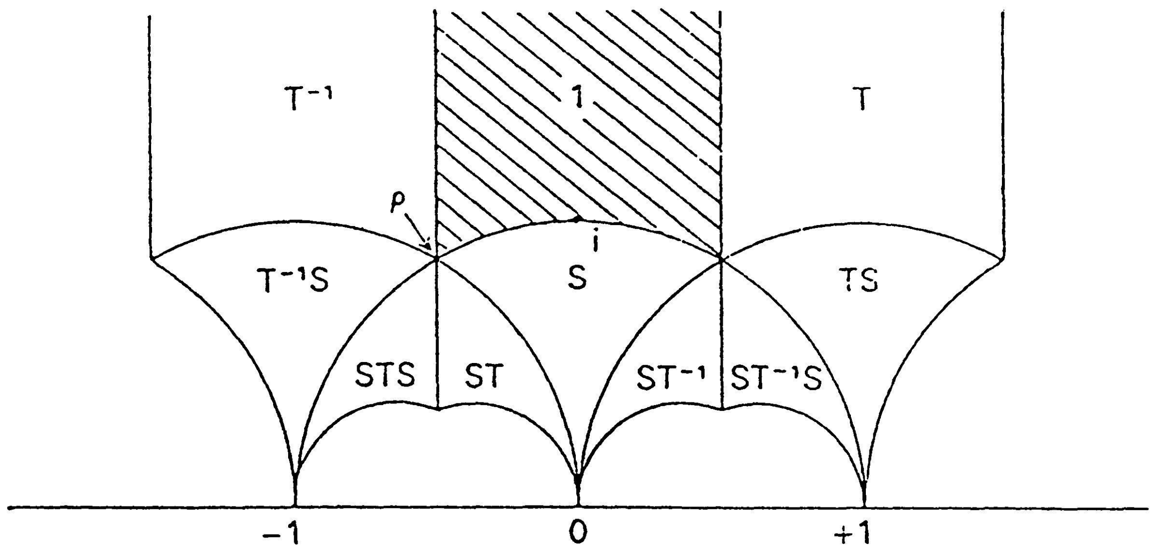
Let us illustrate the principle with the most classical example $\Gamma = SL_2(\mathbf{Z})$ mod. (± 1), $T = \mathscr{H} =$ upper half plane, $F =$ closure of the well-known fundamental domain,

$$F = \left\{ z \in \mathscr{H} \;\middle|\; |z| \geqslant 1, \; -\frac{1}{2} \leqslant Re\, z \leqslant \frac{1}{2} \right\}.$$

(This fits into our general approach since $\mathscr{H}$ is identified with $SO_2(\mathbf{R}) \backslash SL_2(\mathbf{R})$.) Evidently the prerequisites for the theorem are given. Put

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bmod (\pm 1), \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \bmod (\pm 1).$$

We get the well-known picture



of the fundamental domain and its neighbours (taken from [Se2], p. 78). Thus $\tilde{H}$ has 9 generators and a relation $t_{A^{-1}} t_B = t_{A^{-1}B}$ for every pair A, B in the set $\{T, TS, \ldots, T^{-1}S, T^{-1}\}$ such that $FA \cap FB \cap F \neq 0$. This happens only if FA, FB, F meet at ρ or ρT, and we get $2\left[\binom{5}{2}\right] = 20$ relations. It is a puzzle (elementary, but tedious) to derive the presentation

$$\tilde{H} \cong \Gamma/(\pm 1) = \langle S, ST \mid S^2 = (ST)^3 = 1 \rangle = C_2 * C_3 .$$

Admittedly this wouldn't be too easy if the result were not known in advance; but the method works in principle.

To derive presentations along these classical lines is a hard piece of work and has been done only in "small" cases. Swan [Sw] considers $SL_2(R)$, where R is the integral domain of an imaginary quadratic field K. $PSL_2(R)$ is called a Bianchi group, after L. Bianchi who was the first to embark on a systematic study of these groups 100 years ago. (See his *Opere,* Vol. 1, ed. by Bompiani and Sansone, Rome 1952; in particular Sansone's introduction to this part of Bianchi's work on page 185 ff. The fundamental domain below appears on p. 239. This particular case had already been treated by Picard.) Here the geometric unit rank $r(A) = r(SA)$ is 3, the dimension of the space of positive Hermitian forms over $\mathbf{C}$ with discriminant 1, or hyperbolic 3-space. Swan obtains presentations for $K = \mathbf{Q}(\sqrt{-d})$, with $d = 1, 2, 3, 5, 6, 11, 15, 19$. For the sake of illustration, here is the first case: define

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then $SL_2(\mathbf{Z}[i])$ is generated by these matrices, and defining relations are

$$\begin{aligned} TU = UT,\ J^2 = 1,\ J \text{ central},\ L^2 = (TL)^2 = (UL)^2 = (AL)^2 = A^2 \\ = (TA)^3 = (UAL)^3 = J . \end{aligned}$$

If one identifies the hyperbolic 3-space with $\mathbf{R}^3$, a fundamental domain is

$$F = \left\{ (x, y, z) \,\middle|\, x \in \left[-\frac{1}{2}, \frac{1}{2} \right), \ y \in \left[0, \frac{1}{2} \right), \ z > 0, x^2 + y^2 + z^2 \geqslant 1 \right\}.$$

It is interesting to note that the section $y = 0$ of F equals the classical fundamental domain of $SL_2(\mathbf{Z})$ in the upper half plane.

Developing Swan's techniques, Frohman and Fine were able to establish a structure theorem for $SL_2(R)$ which we now describe (following [F]). Let $K = \mathbf{Q}(\sqrt{-d})$ with $d > 0$ a squarefree integer. We exclude here

$d = 1, 2, 3, 7, 11$; these give the euclidean R and require special treatment. The main theorem (6.3.1) is a decomposition as a free product with amalgamation

$$SL_2(R) = PE_2 *_F G(R) .$$

Here (4.8.2),

PE_2 = image in $PSL_2(R)$ of the subgroup generated by elementary matrices
$\cong (\mathbf{Z} \times \mathbf{Z}) *_{\mathbf{Z}} PSL_2(\mathbf{Z})$;

amazingly, this group is independent of d. Likewise, F is explicitly presented (6.3.4) and independent of d. The precise structure of $G(R)$ is not yet fully clear. The monograph [F] contains many more results, e.g. on finite subgroups and normal subgroups. A fact worth mentioning: $SL_2(R)$ contains non-congruence subgroups of finite index ([Se5]).

Some examples of the "analogous" but deeply different case in which R is the integral domain of a *real* quadratic field were treated by Kirchheimer and Wolfart [KW]; these groups are known as Hilbert modular groups. Here, $r(A) = 5$, but $r(SA) = 4$, and in fact $PSL_2(R)$ operates most naturally on a product of two upper halfplanes.

A fundamental domain has been described already by Siegel [S2]. The problems with the boundary become already for small discriminants so considerable as to require the use of machine calculations. The main result of [KW] is as follows: let ε be the fundamental unit of R and put

$$T(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \ a \in R, \ E = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \ S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Then one always has the relations

$$\begin{aligned} T(a)T(b) &= T(b)T(a), a, b \in R, \\ ET(a)E^{-1} &= T(\varepsilon^2 a), a \in R, \\ S^2 = (ES)^2 &= (ST(1))^3 = (EST(\varepsilon))^3 = 1 . \end{aligned}$$

It is shown: if $K = \mathbf{Q}(\sqrt{d})$, $d = 5, 12, 13$, then these relations are defining relations. (For $d = 3$, one additional relation is required.) In [Ki] Kirchheimer treats $SL_2(R)$ for arbitrary totally real R of class number 1; the example $R = \mathbf{Z}[\xi + \xi^{-1}]$, $\xi = e^{2\pi i/7}$ is presented in detail.