9. The Bass unit theorem

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COROLLARY. $S\Gamma/[S\Gamma, S\Gamma]$ *is finite.*

Again, it is classic that the theorem fails for $SL_2(\mathbf{Z})$; for instance, if $n \gg 0$, the verbal subgroup generated by all $6n$-th powers has infinite index ([Ne], p. 143). The corollary holds for $SL_2(\mathbf{Z})$ but fails for torsion free subgroups (which are free). Theorem 7 however carries over to subgroups of finite index.

Two more topics from the general theory of arithmetic groups, which could be specialized to unit groups, are subgroup rigidity and strong approximation. But having promised to keep as near as possible to the unit groups "themselves", we omit this.

9. THE BASS UNIT THEOREM

In his paper [Ba 2] Bass has proved (among other things) a far reaching generalization of Dirichlet's unit theorem which — together with the results of sections 3 and 7 — is certainly one of the strongest general results we have about Γ. The core of the proof is a deep stability theorem from K-theory; we will indicate how it implies the theorem but will say little about its proof. We begin with the relevant definitions. For any ring A, define

$$K_1(A) = \varinjlim GL_n(A)/[GL_n(A), GL_n(A)] ,$$

where the direct limit is taken with respect to the embeddings

$$GL_n(A) \to GL_{n+1}(A), \quad x \to \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} .$$

One may also write

$$K_1(A) = \varinjlim \quad GL_n(A)/\tilde{E}_n(A) ,$$

where $\tilde{E}_n(A)$ is the normal subgroup generated by the elementary matrices; this is Whitehead's lemma. Further, with $K_0(A)$ denoting the Grothendieck group of finitely generated projective A-modules, we put

$$R_0(A) = \mathbf{R} \otimes K_0(A), \ R_1(A) = \mathbf{R} \otimes K_1(A) .$$

Now we turn to algebras and allow A to be semisimple. Let $\Lambda \subset A$ be an order. Any $A_{\mathbf{R}}$-module V (of finite dimension) gives rise to a homomorphism

$$\Lambda \to \mathrm{End}_{\mathbf{R}} V, \text{ hence by functoriality } K_1(\Lambda) \to K_1(\mathrm{End}_{\mathbf{R}} V) .$$

Combining this with

$$\log|\det| : K_1(\mathrm{End}_{\mathbf{R}} V) \to \mathbf{R} ,$$

we obtain a homomorphism

$$f_V : R_1(\Lambda) \to \mathbf{R} ,$$

and it is easy to see that $V \to f_V$ gives a homomorphism

$$f : K_0(A_{\mathbf{R}}) \to R_1(\Lambda)^* = \mathrm{Hom}(R_1(\Lambda), \mathbf{R}) .$$

THEOREM 9. *The sequence*

$$0 \to R_0(A) \to R_0(A_{\mathbf{R}}) \to R_1(\Lambda)^* \to 0 \tag{8}$$

is exact.

The following corollary (which can easily be derived from the sequence) is shown in the course of the proof:

COROLLARY. *Let R be the maximal order of the center K of A. Then*

$$rkK_1(\Lambda) = rkR^{\times} .$$

Dirichlet's theorem arises in the special case $\Lambda = R$, $A = K$. Clearly $rkK_0(K) = 1$, and writing

$$\mathbf{R} \otimes_{\mathbf{Q}} K = \mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$$

as previously, we obtain

$$rkK_1(R) = r_1 + r_2 - 1 .$$

But for Dedekind domains R one knows that

$$rkR^{\times} = rkK_1(R)$$

(see [CR2], §45A). However, Dirichlet's theorem is used in the proof of Theorem 7.

A case of interest is

$$A = \mathbf{Q}G, \Lambda = \mathbf{Z}G \quad (G \text{ is a finite group}) .$$

Here,

$$\begin{aligned} rkK_0(\mathbf{Q}G) &= \text{number of conjugacy classes of cyclic subgroups of } G \\ &= : q(G), \\ rkK_0(\mathbf{R}G) &= \text{number of real conjugacy classes of } G \\ &= : r(g) \end{aligned}$$

(see e.g. [Se 6], 12.4). Thus the theorem gives

$$rk K_1(\mathbf{R}G) = r(G) - q(G) .$$

By Theorem 5 of [Ba 2], $(\mathbf{Z}G)^\times$ is mapped onto a subgroup of finite index in $K_1(\mathbf{Z}G)$. Hence

$$rk\left(\frac{\mathbf{Z}G^\times}{[\mathbf{Z}G^\times, \mathbf{Z}G^\times]}\right) \geqslant r(G) - q(G) .$$

The reader will find it an amusing exercise to work out that for G cyclic this is an equality. However, if $\mathbf{Q}$ is a splitting field for G (as e.g. for G = symmetric group) the inequality tells us nothing new.

Proof of the theorem (sketch). The injectivity of $K_0(A) \to K_0(A_\mathbf{R})$ follows from a wellknown theorem of representation theory (see [CR 1], § 29). Next, it is not difficult to show that $R_0(A) \to R_0(A_\mathbf{R}) \to R_1(\Lambda)^*$ is the zero map: if V is already an A-module, then

$$K_1(\Lambda) \to K_1(\mathrm{End}_\mathbf{R} V) \to \mathbf{R}$$

factors through

$$K_1(\Lambda) \to K_1(\mathrm{End}_\mathbf{Q} V) \to \mathbf{R} ;$$

if $x \in K_1(\Lambda)$ is represented by an element of $Gl_n(\Lambda)$, the image of this element in $K_1(\mathrm{End}_\mathbf{Q} V)$ is an integral unit in a matrix ring over $\mathbf{Q}$, hence has determinant $\pm$ 1. In order to state the crucial lemma, we recall the notion of reduced norm. Assume for the moment that A is simple with center K. Then there exists a splitting field $L \mid K$, for example, a maximal commutative subfield of A, such that

$$L \otimes_K A \cong M_n(L)$$

is isomorphic to a full matrix algebra. Let φ be an isomorphism and define, for $a \in A$,

$$Nr(a) = \det \varphi(1 \otimes a) .$$

One shows that $Nr(a) \in K$ and is independent of the choice of L and φ. The effect of using Nr instead of the usual norm $N_{A \mid K}$ taken with respect to the regular representation of A over K is the elimination of superflous powers; namely, one has

$$N_{A \mid K}(a) = (Nr\, a)^s, \quad \text{where } \dim_K A = s^2 .$$

For semisimple A, Nr is defined componentwise and induces homomorphisms

$$Nr_n : GL_n(\Lambda)/\tilde{E}_n(\Lambda) \to R^\times$$

and in the limit

$$N : K_1(\Lambda) \to R^\times .$$

It is easily seen that Nr_n and N have finite cokernel. Let q be the number of simple factors of A.

LEMMA. *For n sufficiently large, Nr_n and therefore N have finite kernel. Consequently,*

$$rkK_1(\Lambda) = rkR^\times = r_1 + r_2 - q .$$

Taking this for granted, we combine N with

$$\log|\,| : R^\times \to \mathbf{R}^{r_1+r_2},\ c \to (\log|c|_1 \ldots, \log|c|_{r_1+r_2}) .$$

From the lemma, one first derives that every component of

$$\log|N| : K_1(\Lambda) \to \mathbf{R}^{r_1+r_2}$$

has the form f_V for suitable V. Then it follows that there are "enough" linear functionals of this type, that is,

$$f : R_0(A_{\mathbf{R}}) \to R_1(\Lambda)^*$$

is surjective. The exactness of (8) now follows by dimension count.

We cannot say much about the proof of the lemma and refer the reader to [Ba 1]. The main point is that 2 defines a "stable range" for the $\mathbf{Z}$-algebra Λ ([Ba 1], Th. 11.1) which implies, among other things, that for $r > 2$

$$GL_r(\Lambda) = GL_2(\Lambda)E_r(\Lambda)$$

(hence $GL_2(\Lambda) \to K_1(\Lambda)$ surjective) and for $r \geqslant 4$

$$E_r(\Lambda) = [GL_r(\Lambda), GL_r(\Lambda)] .$$

([Ba 1], th. 19.5). Put $Sl_n(\Lambda)$ = Kernel of reduced norm. The above implies that for $n \geqslant 4$ all maps

$$SL_n(\Lambda)/E_n(\Lambda) \to SL_{n+1}(\Lambda)/E_{n+1}(\Lambda) \to SL(\Lambda)/E(\Lambda)$$

are surjective and all these groups are abelian. Since everything is finitely generated, this sequence becomes stationary, i.e.

$$SL_n(\Lambda)/E_n(\Lambda) \cong SL(\Lambda)/E(\Lambda)$$

for all $n \gg 0$. One then has an exact sequence

$$0 \to SL(\Lambda)/E(\Lambda) \to K_1(\Lambda) \to K_1(A) ,$$

and it remains to show that $K_1(\Lambda) \to K_1(A)$ has finite kernel ([Ba 1], 19.12). This implies the lemma.

We have presented Bass' theorem here because it can be viewed as an extension of Dirichlet's unit theorem. For more results on K_1 of orders, we refer the reader to [CR2, Ch. 5]. This chapter also contains a simplified proof of Bass' theorem.

10. What is a unit theorem?

In the search for the — still missing — "basic structure theorem for units of orders" it seems natural to keep Dirichlet's theorem as our landmark; it gives in fact a presentation for all commutative unit groups. However, if we muster the small list of other cases in which explicit presentations have been obtained so far, and if we realize the comparatively elementary character of these examples, we have to admit that going for presentations is somehow utopian. Worse still, it might even be inadequate; as the general insolvability of Dehn's problems shows, we can never be sure that a presentation, obtained somehow, gives us the "right" information. For example, how could the congruence property be checked from a presentation? What then, it will now be objected, is the aim of our research? This is certainly not the place to dwell in considerations in the manner of ordinary language philosophy, but the reader may find it fruitful to ask himself what he means by saying "I know a certain group" or "I know the structure of that group". Surely we know $SL_2(\mathbf{Z})$ better than any other noncommutative unit group, but we will never know everything about it (and hence about groups containing it) because this would include knowledge of all finitely generated groups.

Leaving aside philosophy, let us try to specify what should be expected from a "general unit theorem". Unable, of course, to presume its content, we may be allowed to sketch a list of desiderata.

Let A be simple. The unit theorem should deal with torsion free subgroups of finite index of $S\Gamma$ for arbitrary Λ; such groups may be called "*generic unit groups of A*". The set of generic unit groups is closed under intersections since any two are commensurable. Naively, a unit theorem for A consists in *the definition, in purely group theoretical terms, of a class of groups $\mathscr{C}(A)$ such that almost all generic unit groups of A are members of $\mathscr{C}(A)$.*