

2. Basic material

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **41 (1995)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

2. BASIC MATERIAL

For more details in this section see [A], for example.

(2.1) Recall that G is a compact connected Lie group with maximal torus T , having respective Lie algebras \mathfrak{g} and \mathfrak{t} . The Weyl group is the finite group $W = N/T$, where N is the normalizer in G of T . Since G is compact, there is an $Ad(G)$ -invariant inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} , obtained by averaging any inner product over G . Let \mathfrak{m} be the orthogonal complement of \mathfrak{t} in \mathfrak{g} with respect to this inner product, so

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m} \quad (\text{orthogonal}) .$$

The infinitesimal version of invariance of the inner product is the identity

$$\langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0 ,$$

for $X, Y, Z \in \mathfrak{g}$.

(2.2) The exponential map $\exp: \mathfrak{g} \rightarrow G$ is surjective, since G is compact. This is one of the deeper theorems in a first course on Lie groups. We actually only need this surjectivity for $\exp: \mathfrak{t} \rightarrow T$, which is clear.

The Lie algebra \mathfrak{t} is abelian (the bracket is zero); in fact \mathfrak{t} is a maximal abelian subalgebra of \mathfrak{g} . In particular, no nonzero vector in \mathfrak{m} has zero bracket with all of \mathfrak{t} . Likewise, $Ad(T)$ has no nonzero invariant vectors in \mathfrak{m} .

Now a torus is a topologically cyclic group. That means there exists a *generic element* $t_0 \in T$ whose powers form a dense subgroup of T . It follows that the single operator $Ad(t_0)$ can have no invariants in \mathfrak{m} . Likewise in the group G , it can be shown that a maximal torus is its own centralizer, so the centralizer in G of t_0 is just T . There is a similar notion in the Lie algebra. A *regular element* of \mathfrak{t} is one whose $Ad(G)$ -centralizer is exactly $Ad(T)$. To find one, take any $H_0 \in \mathfrak{t}$ such that $\exp H_0 = t_0$.

(2.3) The group G acts on \mathfrak{g} via Ad , and this induces an action of W on \mathfrak{t} . No element of W acts trivially, and the image of W in $GL(\mathfrak{t})$ is generated by reflections about certain hyperplanes defined as follows.

Since the nontrivial irreducible representations of a torus are given by two dimensional rotations, we have an orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_v$, where each \mathfrak{m}_k is two dimensional and there is a finite set of nonzero linear functionals $\Delta^+ = \{\alpha_1, \dots, \alpha_v\} \subset \mathfrak{t}^*$, called *positive roots* such that for $H \in \mathfrak{t}$, the eigenvalues of $Ad \exp H$ on \mathfrak{m}_i are $\exp(\pm \sqrt{-1} \alpha_i(H))$. We determine the signs as follows. Fix a regular

element $H_0 \in \mathfrak{t}$. We may and shall choose the positive roots so that they take strictly positive values on H_0 . The action of W on \mathfrak{t} is generated by reflections about the kernels of the positive roots.

Since each \mathfrak{m}_i is also preserved by $ad(\mathfrak{t})$, we can choose an orthonormal basis $\{X_i, X_{v+i}\}$ of \mathfrak{m}_i such that, for $H \in \mathfrak{t}$, the matrix of $ad(H)|_{\mathfrak{m}_i}$ with respect to this basis is

$$\begin{pmatrix} 0 & \alpha(H) \\ -\alpha(H) & 0 \end{pmatrix}.$$

Note that the ad -invariance of the inner product \langle , \rangle implies, for all $1 \leq i \leq v$, all $1 \leq j \leq 2v$ and all $H \in \mathfrak{t}$ that

$$\langle H, [X_i, X_j] \rangle = \langle [H, X_i], X_j \rangle = -\alpha_i(H) \langle X_{i+v}, X_j \rangle.$$

By orthonormality, this last pairing can only be nontrivial if $j = i + v$. Hence if $j \neq i + v$, we have $[X_i, X_j] \in \mathfrak{m}$. The same thing happens if $i > v$ and $j \neq i - v$.

On the other hand, for $1 \leq i \leq v$, set $H_i = [X_i, X_{v+i}]$. This is $Ad(T)$ -invariant, so $H_i \in \mathfrak{t}$, and $ad(H_i)\mathfrak{m}_i \subseteq \mathfrak{m}_i$. It follows that the span of X_i, X_{i+v}, H_i is a Lie subalgebra \mathfrak{g}_i of \mathfrak{g} . It is always isomorphic to $\mathfrak{su}(2)$.

3. INVARIANT THEORY

All proofs missing from this section may be found in the textbook [H], the expository article [F], or [Bk].

(3.1) Let

$$\mathcal{S} = \bigoplus_{p=0}^{\infty} \mathcal{S}^p \quad \text{and} \quad \Lambda = \bigoplus_{q=0}^l \quad (l = \dim \mathfrak{t})$$

be the symmetric and exterior algebras on \mathfrak{t}^* , respectively. The adjoint action of W on \mathfrak{t} induces representations of W on \mathcal{S} and Λ by degree-preserving algebra automorphisms. For example, the action of W on Λ^l is multiplication by the *sign character*

$$\varepsilon: W \rightarrow \{\pm 1\} \quad \text{given by} \quad \varepsilon(w) = \det Ad(w)_\mathfrak{t}.$$

Note that $\varepsilon(w)$ is the parity of the number of reflections needed to express $Ad(w)_\mathfrak{t}$.