# 4. Invariant Differential Forms

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*Proof.* The coefficients of $dF_1 \wedge \cdots \wedge dF_q \in (S^{m_1 + \cdots + m_q} \otimes \Lambda^q)^W$ span a $W$-invariant subspace of $S^{m_1 + \cdots + m_q}$, isomorphic to $\Lambda^q$. As in (3.4), these coefficients are harmonic because $m_1 + \cdots + m_q$ is the birthday of $\Lambda^q$, by the multiplicity formula (3.8). $\square$

## 4. Invariant Differential Forms

The ideas in this section go back to E. Cartan and de Rham. For a thorough exposition, see [C-E].

(4.1) Suppose a compact Lie group $G$ acts transitively on a manifold $M$. Let $\tau_g$ be the diffeomorphism of $M$ corresponding to $g \in G$. A differential $p$-form $\omega \in \Omega^p(M)$ is *$G$-invariant* if $\tau_g^* \omega = \omega$. Such a form is determined by its value at any one point of $M$. One shows by averaging that every de Rham cohomology class on $M$ is represented by a $G$-invariant form, and that the subcomplex of invariant forms is preserved by the exterior derivative.

Identify $M = G/K$ where $K$ is the stabilizer of a point $o \in M$. We have an orthogonal decomposition $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{n}$, where $\mathfrak{r}$ is the Lie algebra of $K$. Moreover this decomposition is preserved by $Ad(K)$. For example if $G$ acts on itself by left multiplication then $K = 1$ and $\mathfrak{n} = \mathfrak{g}$. For another example take $M = G/T$, so $K = T$ and $\mathfrak{n} = \mathfrak{m}$. In general, $\mathfrak{n}$ is naturally identified with the tangent space $T_o(M)$, so an invariant form $\tilde{\omega}$ is determined by the skew-symmetric multilinear map

$$\omega = \tilde{\omega}_o : \mathfrak{n} \times \cdots \times \mathfrak{n} \to \mathbf{R} .$$

That is, $\omega \in \Lambda^p \mathfrak{n}^*$. The invariance of $\tilde{\omega}$ under $K$ implies the $Ad(K)$-invariance of $\omega$. Conversely, any element $\omega \in (\Lambda^p \mathfrak{n}^*)^K$ determines a $G$-invariant form $\tilde{\omega}$ on $M$ by the formula

$$\tilde{\omega}_{g \cdot o}\big((d\tau_g)_o X_1, \ldots, (d\tau_g)_o X_p\big) = \omega(X_1, \ldots, X_p) ,$$

for $X_1, \ldots, X_p \in \mathfrak{n}$ and $g \in G$. Thus we may identify the $G$-invariant $p$-forms on $M$ with the space $(\Lambda^p \mathfrak{n}^*)^K$. In this view, the exterior derivative becomes the map $\delta : (\Lambda^p \mathfrak{n}^*)^K \to (\Lambda^{p+1} \mathfrak{n}^*)^K$ given by

$$\delta\omega(X_0, \ldots, X_p) = \frac{1}{p+1} \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j]_{\mathfrak{n}}, X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p) .$$

Here $\hat{}$ means the term is omitted, and $[X_i, X_j]_{\mathfrak{n}}$ is the projection of $[X_i, X_j]$ into $\mathfrak{n}$ along $\mathfrak{r}$. The complex $\{(\Lambda^p \mathfrak{n}^*)^K, \delta\}$ computes the de Rham cohomology of $M$.

(4.2) Assume that $K$ is connected. Since any homomorphism from $K$ to the multiplicative real numbers must be trivial, the determinant is a nonzero element of the one dimensional space $(\Lambda^n \mathfrak{n}^*)^K$, where $n = \dim M$. It follows that $M$ is orientable, and any $G$-invariant $n$-form on $M$ will have nonzero integral over $M$ as soon as it does not vanish at one point.

(4.3) In the case $M = G$ we have the additional symmetry of left and right multiplication by $G \times G$, and every cohomology class contains a bi-invariant representive. The value at $e$ of a bi-invariant form is $Ad(G)$ invariant. Taking the derivative of the condition for $\omega \in \Lambda^p \mathfrak{g}^*$ to be $Ad(G)$-invariant, we find (product rule) that $\omega([X, X_1], X_2, \ldots, X_p) + \cdots + \omega(X_1, \ldots, [X, X_p]) = 0$ for all $X, X_1, \ldots, X_p \in \mathfrak{g}$. It is then not hard to show that this condition implies that $\delta\omega = 0$. Hence all bi-invariant forms are closed. Since $\delta$ commutes with $Ad$, it follows that the de Rham cohomology of $G$ is computed by the complex $(\Lambda \mathfrak{g}^*)^G$, with zero differential. That is, $H(G) \simeq (\Lambda \mathfrak{g}^*)^G$, as graded rings.

## 5. The cohomology of flag manifolds

The *Bruhat Decomposition* is a cell decomposition of the flag manifold $G/T$ into even dimensional cells indexed by elements of the Weyl group $W$. It generalizes the decomposition of the two-sphere (flag manifold of $SU(2)$) into a point and an open disk. The existence of such a decomposition implies that there are no boundary maps in cellular homology, and the cohomology of $H(G/T)$ is nonzero only in even degrees.

It is customary to explain the The Bruhat decomposition in terms of complex groups. For example the flag manifold for $U(n)$ is in fact a homogeneous space for $GL_n(\mathbf{C})$, and the cells can be described as the orbits of certain subgroups of the group of upper triangular complex matrices, which do not live in $U(n)$. We shall, however, describe the cell decomposition of $G/T$ purely in terms of the compact group $G$, using Morse theory. It was Bott, later with Samelson, who first applied Morse theory to the loop space of $G$ from which, combined with results of Borel and Leray, they deduced results on the topology of $G$ and $G/T$. See [BT] for a brief introduction to Morse theory.

(5.1) We need to find a "Morse function" $f$ on $G/T$. This is a smooth real valued function whose Hessian (matrix of second partial derivatives taken in local coordinates) at each critical point has nonzero determinant. How shall we find one? For the unit sphere in $\mathbf{R}^3$ centered at $(0, 0, 0)$, we can take