# 4. $S^1$-FIBRATIONS

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indeed by [$GN_1$, Proposition 5.7], for odd $p$, $\Gamma$ is cyclic of order $2p^2$. The proof there also shows that $2[\gamma_{1,1}]$ is of order $p^2$ and that $p[\gamma_{0,p}]$ is of order 2 in $\Gamma$, so $[\gamma_{2,2+p^2}]$ generates $\Gamma$.

### (D) The projective plane

We saw that when $X$ is aspherical and $\chi(X) \neq 0$ then $\Gamma = 0$ and so our first order invariants vanish. In the presence of non-trivial higher homotopy these invariants need not vanish, despite $\chi(X) \neq 0$, as demonstrated by the example of the real projective plane $X = P^2$.

Write $G \equiv \pi_1(P^2) \cong \mathbf{Z}/2$; denote the generator of $G$ by $t$. Give $P^2$ the customary cell structure consisting of one cell in each of dimensions 0, 1, and 2. The universal cover $\tilde{P}^2$ is naturally identified with $S^2$ and the corresponding cellular chain complex is:

$$C_2(S^2) \xrightarrow{1+t^{-1}} C_1(S^2) \xrightarrow{t^{-1}-1} C_0(S^2) \, .$$

Every element of $\Gamma$ can be represented by a basepoint preserving homotopy $F: P^2 \times I \to P^2$ with $F_0 = F_1 = \mathrm{id}_{P^2}$. We have $\tilde{F}_0 = \tilde{F}_1 = \mathrm{id}_{S^2}$ because the basepoint is preserved. It is easy to verify that the corresponding chain homotopy $\tilde{D}_*: C_*(S^2) \to C_*(S^2)$ is then zero on $C_0(S^2)$ and takes $\tilde{e}_1$ to $\tilde{e}_2 m(1 - t^{-1})$ where $m \in \mathbf{Z}$. By elementary obstruction theory, there exists $F \equiv F^{(m)}$ realizing any $m \in \mathbf{Z}$. In this case $\mathrm{trace}(\tilde{\partial} \otimes \tilde{D}) = (1 + t^{-1}) \otimes m(1 - t^{-1})$ which is homologous to the canonical form $mt^{-1} \otimes tt^{-1} - mt^{-1} \otimes tt^{-2}$. Since $\chi(P^2) = 1 \neq 0$, the Gottlieb group $\eta_{\#}(\Gamma) \equiv \mathscr{G}(P^2) = 0$ and so the derivation $\tilde{\mathrm{X}}_1(P^2)$ is a homomorphism and need not be distinguished from its cohomology class $\tilde{\chi}_1(P^2) \in H^1(\Gamma, HH_1(\mathbf{Z}(\mathbf{Z}/2))) \cong \mathrm{Hom}(\Gamma, HH_1(\mathbf{Z}(\mathbf{Z}/2)))$. It follows that

$$\tilde{\chi}_1(P^2)\,([F^{(m)}]) = (m, -m) \in \mathbf{Z}/2 \oplus \mathbf{Z}/2 \cong HH_1(\mathbf{Z}(\mathbf{Z}/2)) \, .$$

In particular, when $m$ is odd $\tilde{\chi}_1(P^2)\,([F^{(m)}]) \neq 0$. On the other hand, this shows $\chi_1(P^2) = 0$.

## 4. $S^1$-Fibrations

In this section we investigate the first order Euler characteristic of the total space of an orientable Serre fibration with $S^1$-fiber.

Let $S^1 \to X \xrightarrow{\pi} B$ be an orientable Serre fibration where $B$ is a (not necessarily finite) connected CW complex and $X$ has the homotopy type of a finite complex. By classical obstruction theory, fiber homotopy equivalence classes of orientable $S^1$-fibrations over a CW complex $B$ are classified by the integral cohomology group $H^2(B;\mathbf{Z})$. Given an element $e \in H^2(B;\mathbf{Z}) \cong [B, \mathbf{C}P^\infty]$ one obtains a principal $U(1)$-bundle over $B$ by pulling back, via a continuous map $B \to \mathbf{C}P^\infty$ representing $e$, the $U(1)$-bundle associated to the canonical complex line bundle over the infinite dimensional complex projective space $\mathbf{C}P^\infty$. Thus we can assume, without loss of generality, that $S^1 \to X \xrightarrow{\pi} B$ is a principal $U(1)$-bundle. In particular, there is a free $U(1)$-action on $X$ which we will write as $\Phi: X \times S^1 \to X$. Let $\tau \in \Gamma \equiv \pi_1(\mathscr{E}(X), 1)$ be the element represented by $\Phi$ ($\Phi = \Phi^\tau$ in the notation of §1). For any coefficient ring $R$, let $\{r\} \in H_1(X; R)$ denote the image of $\tau$ under the composite:

$$\Gamma \xrightarrow{\eta} \pi_1(X) \to H_1(X) \to H_1(X; R)\ .$$

Also, let $e_R$ be the image of the element $e \in H^2(B;\mathbf{Z})$ which classifies $S^1 \to X \xrightarrow{\pi} B$ under the homomorphism $H^2(B;\mathbf{Z}) \to H^2(B;R)$.

LEMMA 4.1. *If* $\mathbf{F}$ *is a field, then* $\{\tau\} \in H_1(X;\mathbf{F})$ *is non-zero if and only if* $e_{\mathbf{F}} = 0$.

*Proof.* Consider the Gysin homology sequence for the fibration $S^1 \to X \xrightarrow{\pi} B$:

$$\cdots \to H_2(B;\mathbf{F}) \xrightarrow{e_{\mathbf{F}}\cap} H_0(B;\mathbf{F}) \xrightarrow{\theta_0} H_1(X;\mathbf{F}) \xrightarrow{\pi_*} H_1(B;\mathbf{F}) \to 0\ .$$

Since $H_2(B;\mathbf{F}) \xrightarrow{e_{\mathbf{F}}\cap} H_0(B;\mathbf{F}) \cong \mathbf{F}$ is just evaluation of the cohomology class $e_{\mathbf{F}}$ on homology, $\theta_0$ is non-zero if and only if $e_{\mathbf{F}} = 0$. Let $\upsilon \in X$ be a basepoint and let $\{\pi(\upsilon)\} \in H_0(B;\mathbf{F})$ be the generator determined by the inclusion of $\pi(\upsilon)$ into $B$. The fact that $\theta_0(\{\upsilon\}) = \{\tau\}$ follows from the naturality of the Gysin sequence homology sequence, by mapping the Gysin sequence of the trivial fibration $S^1 \to S^1 \to \pi(\upsilon)$, via the homomorphism induced by inclusion, into the Gysin sequence for $S^1 \to X \xrightarrow{\pi} B$. □

THEOREM 4.2. *Let* $\mathbf{F}$ *be a field. If* $e_{\mathbf{F}} \neq 0$ *then* $\chi_1(X;\mathbf{F})(\tau) = 0$. *If* $e_{\mathbf{F}} = 0$ *then* $H_*(B;\mathbf{F})$ *is finite dimensional over* $F$ *and* $\chi_1(X;\mathbf{F})(\tau) = -\chi(B;\mathbf{F})\{\tau\}$ *where* $\chi(B;\mathbf{F}) = \sum_{i\geqslant 0}(-1)^i \dim_{\mathbf{F}} H_i(B;\mathbf{F})$.

*Proof.* In this proof, all homology and cohomology groups will have coefficients in the field $\mathbf{F}$. Since $B$ is the orbit space of the $U(1)$-action on $X$ given by $\Phi$, there is a commutative square:

$$\begin{array}{ccc} X \times S^1 & \xrightarrow{\Phi} & X \\ \pi \times \mathrm{id} \downarrow & & \pi \downarrow \\ B \times S^1 & \xrightarrow{p} & B \end{array}$$

where $p\colon B \times S^1 \to B$ is projection. This square induces a commutative ladder mapping the Gysin homology sequence of $S^1 \to X \times S^1 \xrightarrow{\pi \times \mathrm{id}} B \times S^1$ to the Gysin homology sequence of $S^1 \to X \xrightarrow{\pi} B$:

$$\begin{array}{ccccccc}
H_i(B \times S^1) & \xrightarrow{\theta'} & H_{i+1}(X \times S^1) & \xrightarrow{(\pi \times \mathrm{id})_*} & H_{i+1}(B \times S^1) & \to & H_{i-1}(B \times S^1) \\
{\scriptstyle p_*}\downarrow & & {\scriptstyle \Phi_*}\downarrow & & {\scriptstyle p_*}\downarrow & & {\scriptstyle p_*}\downarrow \\
H_i(B) & \xrightarrow{\theta} & H_{i+1}(X) & \xrightarrow{\pi_*} & H_{i+1}(B) & \xrightarrow{e_{\mathbf{F}} \cap} & H_{i-1}(B)
\end{array}$$

For each integer $0 \leqslant i \leqslant \dim X$ choose a basis $\{b_1^i, \ldots, b_{\beta_i}^i\}$ for $H_i(X)$ such that for some integer $m_i \leqslant \beta_i \{b_{m_i+1}^i, \ldots, b_{\beta_i}^i\}$ is a basis for the kernel of $\pi_*\colon H_i(X) \to H_i(B)$. The corresponding dual basis for $H^i(X)$ will be denoted by $\{\bar{b}_1^i, \ldots, \bar{b}_{\beta_i}^i\}$. Since we are using coefficients in a field, we make the identifications $H_*(B \times S^1) \cong H_*(B) \otimes H_*(S^1)$ and $H_*(X \times S^1) \cong H_*(X) \otimes H_*(S^1)$ via the natural isomorphism given by the homology exterior product. Let $u \in H_1(S^1)$ be the generator determined by the standard orientation of $S^1$. Using Definition $B_1$,

$$\chi_1(X; \mathbf{F})(\tau) = \sum_{k \geqslant 0} (-1)^{k+1} \sum_{j=1}^{\beta_k} \bar{b}_j^k \cap \Phi_*(b_j^k \otimes u) .$$

Consider $b_j^i \otimes u \in H_{i+1}(X \times S^1)$ where $m_i + 1 \leqslant j \leqslant \beta_i$. Since $b_j^i$ lies in $\ker \pi_*$, the exactness of the Gysin sequence implies that $b_j^i \otimes u = \theta'(c \otimes u)$ for some $c \in H_i(B)$. Consequently,

$$\Phi_*(b_j^i \otimes u) = \Phi_*(\theta'(c \otimes u)) = \theta(p_*(c \otimes u)) = 0$$

because $p_*(c \otimes u) = 0$. It follows that

$$\chi_1(X; \mathbf{F})(\tau) = \sum_{k \geqslant 0} (-1)^{k+1} \sum_{j=1}^{m_k} \bar{b}_j^k \cap \Phi_*(b_j^k \otimes u) . \tag{4.3}$$

For each $k$, the set $\{\pi_*(b_1^k), \ldots, \pi_*(b_{m_k}^k)\}$ is a basis for the image of $\pi_*\colon H_k(X) \to H_k(B)$. Extend this set (in any manner) to basis for $H_k(B)$ and let $\{\overline{\pi_*(b_1^k)}, \ldots, \overline{\pi_*(b_{m_k}^k)}\}$ denote the corresponding portion of the dual basis for $H^k(B)$. Then $\bar{b}_j^k = \pi^*(\overline{\pi_*(b_j^k)}), 0 \leqslant j \leqslant m_k$. Consider the commutative diagram:

$$\begin{array}{ccc}
H^k(B \times S^1) & \xrightarrow{(\pi \times \mathrm{id})^*} & H^k(X \times S^1) \\
{\scriptstyle p^*}\uparrow & & {\scriptstyle \Phi^*}\uparrow \\
H^k(B) & \xrightarrow{\pi^*} & H^k(X) .
\end{array}$$

Then, for $0 \leqslant j \leqslant m_k$,

$$\begin{aligned}
\bar{b}_j^k \cap \Phi_*(b_j^k \otimes u) &= \Phi_*(\Phi^*(\bar{b}_j^k) \cap (b_j^k \otimes u)) \\
&= \Phi_*(\Phi^*(\pi^*\overline{(\pi_*(b_j^k))}) \cap (b_j^k \otimes u)) \\
&= \Phi_*((\pi \times \mathrm{id})^*(p^*\overline{(\pi_*(b_j^k))}) \cap (b_j^k \otimes u)) \\
&\quad \text{using the above diagram} \\
&= \Phi_*((\bar{b}_j^k \otimes 1) \cap (b_j^k \otimes u)) \\
&= \Phi_*((\bar{b}_j^k \cap b_j^k) \otimes u) = \Phi_*(\{v\} \otimes u) = \{\tau\}
\end{aligned}$$

where $\{v\}$ is the natural generator of $H_0(X)$ determined by the inclusion of the basepoint $v$ into $X$. From the proof of Lemma 4.1, $\Phi_*(\{v\} \otimes u) = \{\tau\}$. Substituting the above computation into Formula 4.3 yields $\chi_1(X;\mathbf{F})(\tau) = (\sum_{k \geqslant 0}(-1)^{k+1}m_k)\{\tau\}$. If $e_{\mathbf{F}} \neq 0$ then Lemma 4.1 implies that $\{\tau\} = 0$ and so $\chi_1(X;\mathbf{F})(\tau) = 0$. Thus the conclusion of the theorem is valid in this case. If $e_{\mathbf{F}} = 0$ then from the portion

$$H_k(X) \xrightarrow{\pi_*} H_k(B) \xrightarrow{e_{\mathbf{F}} \cap} H_{k-2}(B)$$

of the Gysin homology sequence we deduce that $\pi_*$ is onto and consequently $m_k = \dim_{\mathbf{F}} H_k(B, \mathbf{F})$. Thus $\dim_{\mathbf{F}} H_*(B, \mathbf{F})$ is finite and $\sum_{k \geqslant 0}(-1)^{k+1}m_k = -\chi(B;\mathbf{F})$. $\square$

Theorem 4.2 can be used to recalculate $\chi_1(X;\mathbf{F})$ in Examples 3.8 and 3.9.

Next, we consider integer coefficients. Suppose that $S^1 \to X \xrightarrow{\pi} B$ is a smooth orientable $U(1)$-bundle over a smooth, closed, oriented manifold $B$. Let $\lambda$ be the one dimensional subbundle of the tangent bundle of $X$ consisting of vectors which are tangent to the circle fibers and let be $\nu$ be a complementary bundle to $\lambda$. Then $\nu \cong \pi^*(T_B)$ where $T_B$ is the tangent bundle of $B$. Let $[B] \in H_n(B;\mathbf{Z})$ be the fundamental class of $B$ where $n = \dim B$. The Euler class, $\mathrm{Eul}(\nu) \in H^n(X;\mathbf{Z})$, is given by

$$\mathrm{Eul}(\nu) = \mathrm{Eul}(\pi^*(T_B)) = \pi^*(\mathrm{Eul}(T_B)) = \chi(B)\pi^*([B]^*)$$

where $[B]^* \in H^n(B;\mathbf{Z})$ is the generator determined by the condition $[B]^*([B]) = 1$; see [MS, Corollary 11.12]. The Gysin homology sequence for $S^1 \to X \xrightarrow{\pi} B$ determines a fundamental class for $X$; $[X] \in H_{n+1}(X)$ is the image of $[B]$ under the homomorphism $\theta_n: H_n(B;\mathbf{Z}) \to H_{n+1}(X;\mathbf{Z})$. For any closed oriented $m$-dimensional manifold $M$, let $\mathrm{PD}_M: H^i(M) \to H_{m-i}(M)$ be the Poincaré duality isomorphism explicitly given by $\mathrm{PD}_M(x) = (-1)^{i(m-i)}x \cap [M]$ where $x \in H^i(M)$ and $[M] \in H_m(M)$ is the fundamental class ($(-1)^{i(m-i)}$ appears because of our use of Dold's sign conventions). An immediate consequence of Theorem 3.1 of [$GN_2$] is the following computation of $\chi_1(X)$ (with integer coefficients):

THEOREM 4.4. $\chi_1(X)(\tau) = -\mathrm{PD}_X(\mathrm{Eul}(\nu))$. □

THEOREM 4.5. *Under the above hypotheses,* $\chi_1(X)(\tau) = -\chi(B)\{\tau\}$.

*Proof.* There is a Poincaré duality isomorphism between the Gysin homology sequence and the Gysin cohomology sequence, a portion of which is shown below:

$$\begin{array}{ccccc} H_0(B;\mathbf{Z}) & \xrightarrow{\theta_0} & H_1(X;\mathbf{Z}) & \xrightarrow{\pi_*} & H_1(B;\mathbf{Z}) \\ \mathrm{PD}_B \uparrow & & \mathrm{PD}_X \uparrow & & \mathrm{PD}_B \uparrow \\ H^n(B;\mathbf{Z}) & \xrightarrow{\pi^*} & H^n(X;\mathbf{Z}) & \rightarrow & H^{n-1}(B;\mathbf{Z}) \end{array}$$

Let $\upsilon \in X$ be a basepoint, and let $\{\pi(\upsilon)\} \in H_0(B;\mathbf{Z})$ be the generator determined by the inclusion of $\pi(\upsilon)$ into $B$. From the above diagram, $\mathrm{PD}_X(\pi^*([B]^*)) = \theta_0(\{\pi(\upsilon)\})$. Also, from the proof of Lemma 4.1, $\theta_0(\{\pi(\upsilon)\}) = \{\tau\}$. Thus $\mathrm{PD}_X(\mathrm{Eul}(\nu)) = \chi(B)\{\tau\}$. Regarding the free $U(1)$-action on $X$ as a flow, we can now invoke Theorem 4.4 to conclude that $\chi(B)\{\tau\} = -\chi_1(X)(\tau)$. □

*Example 4.6.* Let $\Sigma_g$ be a closed oriented surface of genus $g > 1$ and let $L_n$ be a complex line bundle over $\Sigma_g$ with Chern number $n$. Let $M_{n,g}$ be the total space of the $U(1)$-bundle associated to $L_n$. Then $M_{n,g}$ is a closed oriented aspherical 3-manifold which fibers over $\Sigma_g$. The center of $\pi_1(M_{n,g})$ is the infinite cyclic group generated by $\tau$ (represented by a circle fiber); the image, $\{\tau\}$, of $\tau$ in $H_1(M_{n,g}) \cong \mathbf{Z}^{2g} \oplus \mathbf{Z}/n$ generates the $\mathbf{Z}/n$ summand. By Theorem 4.5, $\chi_1(M_{n,g}): \mathbf{Z} \to H_1(M_{n,g})$ is given by $\chi_1(M_{n,g})(\tau) = (2g-2)\{\tau\}$.

Let $T^n$, where $n > 1$, be the $n$-torus (i.e. the $n$-fold product of copies of $U(1)$). Let $X$ be a closed oriented smooth manifold and let $\rho: T^n \times X \to X$ be a smooth free action of $T^n$. This action defines a homomorphism $\bar{\rho}: T^n \to \mathrm{Diff}(X)$ where $\mathrm{Diff}(X)$ is the diffeomorphism group of $X$. Let $\Gamma_\rho \subset \Gamma$ be the image of the composite:

$$\pi_1(T^n, 1) \xrightarrow{\bar{\rho}_\#} \pi_1(\mathrm{Diff}(X), \mathrm{id}) \rightarrow \pi_1(\mathscr{E}(X), \mathrm{id}) = \Gamma .$$

PROPOSITION 4.7. *The restriction of* $\chi_1(X): \Gamma \to H_1(X)$ *to* $\Gamma_\rho$ *is the zero homomorphism.*

*Proof.* Since $n > 1$, if $T \subset T^n$ is a circle subgroup then $\chi(X/T) = 0$. Applying Theorem 4.5 to the bundle $T \to X \to X/T$ yields the conclusion. $\square$

COROLLARY 4.8. *If* $n > 1$ *then* $\chi_1(T^n): \mathbf{Z}^n \to \mathbf{Z}^n$ *is zero.* $\square$

## 5. A HIGHER ANALOG OF GOTTLIEB'S THEOREM

Let $G$ be a group of type $\mathscr{F}$. Gottlieb's theorem (see Propositions 1.3 and 2.4) asserts that if $\chi(G) \neq 0$ then $Z(G)$, the center of $G$, is trivial. We prove an analogous theorem for $\chi_1(G; \mathbf{Q})$: if $\chi_1(G; \mathbf{Q}) \neq 0$ then the center of $G$ is infinite cyclic provided $G$ satisfies an extra hypothesis (explained below) related to the Bass Conjecture; see Proposition 5.2 and Theorem 5.4.

Throughout this section $R$ will be a commutative ground ring. Let $S$ be any associative $R$-algebra with unit. The Hochschild homology group $HH_0(S)$ is the $R$-module $S/[S, S]$ where $[S, S]$ is the $R$-submodule of $S$ generated by $\{ab - ba \mid a, b \in S\}$; see §2. Recall that $K_0(S)$ is the abelian group $F/A$ where $F$ is the free abelian group generated by the set of all isomorphism classes $[M]$ of finitely generated projective right $S$-modules $M \subset \bigoplus_{i=1}^{\infty} S$ and $A$ is the subgroup of $F$ generated by relations of the form $[M_1 \oplus M_2] - [M_1] - [M_2]$. Since a finitely generated projective module is the image of a finitely generated free module under an idempotent homomorphism, each element of $K_0(S)$ can be represented by an idempotent matrix over $S$. The *Hattori-Stallings* trace $T_0: K_0(S) \to HH_0(S)$ is defined as follows. Let $A: M \to M$ be an idempotent endomorphism of a free, finitely generated right $S$-module $M$ representing $x \in K_0(S)$. If $[A]$ is the matrix of $A$ with respect to a given basis for $M$ then $T_0(x)$ is defined to be $T_0([A]) \in HH_0(S)$.

Consider the groupring, $RG$, of a group $G$ over $R$. Then $HH_0(RG)$ is naturally isomorphic to the free $R$-module generated by $G_1$, the set of conjugacy classes of $G$ (see §2 for an explanation in the case $R = \mathbf{Z}$). Recall that for $g \in G$ we write $C(g) \in G_1$ for the conjugacy class of $g$, $HH_0(RG)_{C(g)}$ for the summand of $HH_0(RG)$ corresponding to $C(g)$ and $x_{C(g)}$ for the $C(g)$-component of $x \in HH_0(RG)$. Also write $HH_0(RG) = HH_0(RG)_{C(1)} \oplus HH_0(RG)'$ where $1 \in G$ is the identity element of $G$, and $HH_0(RG)'$ is the direct sum of the remaining summands. The augmentation homomorphism $\varepsilon: RG \to R$ induces a homomorphism $\varepsilon_*: HH_0(RG) \to HH_0(R) = R$.