# 2. Realization of cubic forms

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where the 'degree' $d$ corresponds to the cubic form. Such a 4-tuple is admissible iff $d(2x + \bar{W})^3 \equiv (p + 24\bar{T}) \cdot (2x + \bar{W}) \pmod{48}$ holds for every integer $x$. This is equivalent to $p \equiv 4d \pmod{24}$ if $\bar{W} = 0$, and to $p \equiv d + 24\bar{T} \pmod{48}$ with $d \equiv 0 \pmod 2$ if $\bar{W} \neq 0$.

Two admissible 4-tuples $(\bar{W}, \bar{T}, d, p)$ and $(\bar{W}', \bar{T}', d', p')$ are equivalent iff $\bar{W}' = \bar{W}, \bar{T}' = \bar{T}$ and $(d', p') = \pm(d, p)$. Taking the degree $d$ non-negative, we find:

PROPOSITION 1. *There is a 1-1 correspondence between oriented homeomorphism types of cores* $X_0$ *with* $b_2(X_0) = 1$, *and 4-tuples* $(\bar{W}, \bar{T}, d, p)$, *normalized so that* $d \geqslant 0$, *and* $p \geqslant 0$ *if* $d = 0$, *which satisfy* $p \equiv 4d \pmod{24}$ *if* $\bar{W} = 0$, *and* $d \equiv 0 \pmod 2$, $p \equiv d + 24\bar{T} \pmod{48}$ *if* $\bar{W} \neq 0$.

In order to classify the associated homotopy types we first have to determine the subgroup $U_F$ associated to a given cubic form $F$. By definition we find $U_F = 0$ if $d \equiv 0 \pmod 2$, $U_F = \mathbf{Z}_{/2}$ if $d \equiv 1 \pmod 2$. Two normalized 4-tuples $(\bar{W}, \bar{T}, d, p)$ and $(\bar{W}', \bar{T}', d', p')$ are weakly equivalent iff $d' = d, \bar{W}' = \bar{W}$, and $p + 24\bar{T} \equiv p' + 24\bar{T}' \pmod{48}$ if $d \equiv 0 \pmod 2$, $p \equiv p' \pmod{24}$ if $d \equiv 1 \pmod 2$.

Putting everything together, we find a single oriented homotopy type for every odd degree $d \geqslant 0$, which is necessarily spin, and 3 oriented homotopy types for every even degree $d \geqslant 0$; one of these 3 types has $\bar{W} \neq 0$, the other two are spin, and they are distinguished by $p + 24\bar{T} \pmod{48}$ i.e. $p \equiv 4d \pmod{48}$, or $p \equiv 4d + 24 \pmod{48}$.

## 2. REALIZATION OF CUBIC FORMS

In the previous section the (homotopy) topological classification of 1-connected, closed, oriented, 6-dimensional manifolds with torsion-free homology has been transformed into an arithmetical moduli problem: to describe the sets of (weak) equivalence classes of admissible systems of invariants. In this section we begin to investigate the latter problem; we give a simple criterion for the realizability of cubic forms by smooth manifolds, and we describe, at least in principle, the classification of homotopy types of manifolds with a given cohomology ring.

## 2.1 Cohomology rings of 6-manifolds

Let $(r, H, w, \tau, F, p)$ be a system of invariants as in section 1; recall that it is admissible iff for every $W \in H$, $T \in H^{\vee}$ with $\bar{W} = w \pmod 2$, $\bar{T} \equiv \tau \pmod 2$ the following congruence holds:

$$(*) \qquad W^3 \equiv (p + 24T)(W) \pmod{48} .$$

Lemma 1. *$(r, H, w, \tau, F, p)$ is admissible if and only if there exist $W_{\circ} \in H$, $T_{\circ} \in H^{\vee}$ with $\bar{W}_{\circ} \equiv w \pmod 2$, $\bar{T}_{\circ} \equiv \tau \pmod 2$, such that*

*i)* $W_{\circ}^3 \equiv (p + 24T_{\circ})(W_{\circ}) \pmod{48}$

*ii)* $p(x) \equiv 4x^3 + 6x^2 W_{\circ} + 3x W_{\circ}^2 \pmod{24}$ $\forall x \in H.$

*Proof.* Obvious since the set of integral lifts of $w$ is a coset $W_{\circ} + 2H$.

Definition 3. *Let $F \in S^3 H^{\vee}$ be a symmetric trilinear form on a finitely generated free abelian group $H$. An element $W \in H$ is characteristic for $F$ iff*

$$(**) \qquad x \cdot y \cdot (x + y + W) \equiv 0 \pmod 2 \ \ \forall x, y \in H .$$

Lemma 2. *$W \in H$ is a characteristic element for $F \in S^3 H^{\vee}$ if and only if the function $l_W : H \to \mathbf{Z}$, $l_W(x) := 4x^3 + 6x^2 W + 3x W^2$ is linear in $x$ modulo 24.*

*Proof.* $l_W(x + y) = l_W(x) + l_W(y) + 12(x^2 y + x y^2 + x y W)$, whence the assertion.

The existence of characteristic elements is a necessary and sufficient condition for a cubic form $F \in S^3 H^{\vee}$ to be realizable by a manifold. In fact, we have:

Proposition 2. *A given cubic form $F \in S^3 H^{\vee}$ on a finitely generated free abelian group $H$ is realizable as cup-form of a 1-connected, closed, oriented, 6-dimensional manifold with torsion-free homology if and only if it possesses a characteristic element.*

*Proof.* If $(r, H, w, \tau, F, p)$ is an admissible system of invariants, and $W_{\circ} \in H$ any integral lift of $w$, then we have $p(x) \equiv 4x^3 + 6x^2 W_{\circ} + 3x W_{\circ}^2 \pmod{24}$ $\forall x \in H$, i.e. the function $l_{W_{\circ}} : H \to \mathbf{Z}$ is linear modulo 24, and $W_{\circ}$ is therefore characteristic for $F$. Conversely, suppose $W_{\circ} \in H$ is a characteristic element for a cubic form $F \in S^3 H^{\vee}$; let $w := \bar{W}_{\circ} \pmod 2$, $r := 0$.

By the main lemma we have to construct linear forms $p, T \in H^\vee$, such that

i) $W_\circ^3 \equiv (p + 24T)(W_\circ) \pmod{48}$

ii) $p(x) \equiv 4x^3 + 6x^2 W_\circ + 3x W_\circ^2 \pmod{24}$ $\forall x \in H$.

The function $l_{W_\circ}: H \to \mathbf{Z}, l_{W_\circ}(x) = 4x^3 + 6x^2 W_\circ + 3x W_\circ^2$ is linear modulo 24 since $W_\circ$ is a characteristic element for $F$: we therefore choose a linear form $p_\circ \in H^\vee$ with $p_\circ(x) \equiv l_{W_\circ}(x) \pmod{24}$ $\forall x \in H$. Substituting $x = W_\circ$ we find $p_\circ(W_\circ) \equiv 13 W_\circ^3 \pmod{24}$; but since $W_\circ$ is characteristic we have $W_\circ^3 \equiv 0 \pmod 2$, thus $p_\circ(W_\circ) \equiv W_\circ^3 \pmod{24}$. Write $p_\circ(W_\circ) = W_\circ^3 + 24k$ for some $k \in \mathbf{Z}$.

case 1) $k \equiv 0 \pmod 2$: define $p := p_0$, $T := 0$.

case 2) $k \equiv 1 \pmod 2$: we must find a linear form $T_\circ \in H^\vee$ with $T_\circ(W_\circ) \equiv 1 \pmod 2$; clearly this can be done if and only if $W_\circ$ is not divisible by 2. If $W_\circ$ were divisible by 2, $W_\circ = 2V_\circ$ for some $V_\circ \in H$, then $2p_\circ(V_\circ) = p_\circ(W_\circ) = W_\circ^3 + 24k = 8V_\circ^3 + 24k$ would give $p_\circ(V_\circ) = 4V_\circ^3 + 12k$; then, using $p_\circ(V_\circ) \equiv 4V_\circ^3 + 6V_\circ^2 W_\circ + 3V_\circ W_\circ^2 \equiv 4V_\circ^3 \pmod{24}$ we would find $k \equiv 0 \pmod 2$, which is not the case by assumption.

This shows that $F \in S^3 H^\vee$ is realizable by a topological manifold with Pontrjagin class $p_\circ$ and non-vanishing triangulation obstruction $\tau_\circ := \bar{T}_\circ \pmod 2$. In order to realize $F$ by a smooth manifold, one can take $p := p_\circ + 24T_\circ$, and $\tau := 0$.

REMARK 3. The topological counterpart of the existence of a characteristic element for a given cubic form $F \in S^3 H^\vee$ is the existence of a mod-2 Steenrod-algebra structure, which is a necessary condition for a ring to be a cohomology ring.

The existence and the classification of characteristic elements for a given cubic form is essentially a linear algebra problem over $\mathbf{Z}_{/2}$. To see this, let $F \in S^3 H^\vee$ be a fixed cubic form on a finitely generated free abelian group $H$. Associated with $F$ we have a linear map $F^t: H \to S^2 H^\vee$ sending an element $h \in H$ to the bilinear form $F^t(h): H \otimes H \to \mathbf{Z}, (x, y) \to x \cdot y \cdot h$. Let $\bar{H} := H/_{2H}$. $\bar{F} \in S^3 \bar{H}^\vee$ be the reductions of $H$ and $F$ modulo 2, and let $- : H \to \bar{H}$ be the natural epimorphism. The symmetric trilinear form $\bar{F}$ on the $\mathbf{Z}_{/2}$-module $\bar{H}$ defines a natural symmetric bilinear form $q_{\bar{F}} \in S^2 \bar{H}^\vee$ given by $q_{\bar{F}}(\bar{x}, \bar{y}) := \bar{x} \cdot \bar{y} \cdot (\bar{x} + \bar{y})$.

LEMMA 3. *$F \in S^3 H^\vee$ admits characteristic elements if and only if $q_{\bar{F}}$ lies in the image of $\bar{F}^t \in Hom_{\mathbf{Z}}(H, S^2 \bar{H}^\vee)$. The set of all characteristic elements for $F$ is a coset of the form* $W_\circ + \mathrm{Ker}(\bar{F}^t)$.

*Proof.* $W_\circ$ is characteristic for $F$ if and only if $q_{\bar{F}} = \bar{F}^t(W_\circ)$.

In terms of a $\mathbf{Z}$-basis $\{e_1, \ldots, e_b\}$ for $H$ the condition $q_{\bar{F}} \in \operatorname{Im}(\bar{F}^t)$ translates into a simple rank condition over $\mathbf{Z}_{/2}$: the $\mathbf{Z}_{/2}$-rank of the $b \times \binom{b+1}{2}$-matrix $A$ representing $\bar{F}^t$ must be equal to the $\mathbf{Z}_{/2}$-rank of the matrix $A$ extended by the column $(\bar{e}_i \cdot \bar{e}_j \cdot (\bar{e}_i + \bar{e}_j))_{1 \leqslant i \leqslant j \leqslant b}$

EXAMPLE 3. Let $H = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$ be free of rank 2, $F \in S^3 H^\vee$ given by $e_1^3 = a$, $e_1^2 e_2 = b$, $e_1 e_2^2 = c$, $e_2^3 = d$ with $a, b, c, d \in \mathbf{Z}$. The rank condition becomes

$$rk_2 \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \\ \bar{b} & \bar{c} \end{bmatrix} = rk_2 \begin{bmatrix} \bar{a} & \bar{b} & \bar{0} \\ \bar{c} & \bar{d} & \bar{0} \\ \bar{b} & \bar{c} & \overline{b+c} \end{bmatrix}$$

## 2.2 HOMOTOPY TYPES WITH A GIVEN COHOMOLOGY RING

Our next task is to describe the set of oriented homotopy types of 1-connected, closed, oriented, 6-dimensional manifolds with a fixed torsion-free cohomology ring.

From Žubr's classification theorem we know that in algebraic terms this means the following: fix a non-negative integer $r_\circ$, a finitely generated free abelian group $H_\circ$, and a symmetric trilinear form $F_\circ \in S^3 H_\circ^\vee$ which admits characteristic elements.

Let $\mathscr{M}(r_\circ, H_\circ, F_\circ)$ be the set of 1-connected, closed, oriented, 6-dimensional manifolds $X$ with $b_3(X) = 2r_\circ$, such that there exists an isomorphism $\alpha : H_\circ \to H^2(X, \mathbf{Z})$ with $\alpha^* F_X = F_\circ$. Denote by $\operatorname{Aut}(F_\circ)$ the subgroup of $\mathbf{Z}$-automorphisms of $H_\circ$ which leave $F_\circ \in S^3 H_\circ^\vee$ invariant; $\operatorname{Aut}(F_\circ)$ acts on pairs $(w, [l]) \in \bar{H}_\circ \times H_\circ^\vee /_{48 H_\circ^\vee} /_{U_{F_\circ}}$ in a natural way:

$$\gamma \cdot (w, [l]) := (\gamma(w), (\gamma^{-1})^* [l]) .$$

Let ${}_{\operatorname{Aut}(F_\circ)} \backslash \bar{H}_\circ \times H_\circ^\vee /_{48 H_\circ^\vee} /_{U_{F_\circ}}$ be the set of $\operatorname{Aut}(F_\circ)$-orbits.

A manifold $X$ in $\mathscr{M}(r_\circ, H_\circ, F_\circ)$ and an isomorphism $\alpha : H_\circ \to H^2(X, \mathbf{Z})$ with $\alpha^* F_X = F_\circ$ yields a well-defined $\operatorname{Aut}(F_\circ)$-orbit:

$$(\alpha^{-1}(w_2(X)), \alpha^* [p_1(X) + 24T]) \text{ (modulo } \operatorname{Aut}(F_\circ)) ,$$

where $T \in H^4(X, \mathbf{Z})$ is an arbitrary integral lifting of $\tau(X) \in H^4(X, \mathbf{Z}_{/2})$.

The set of oriented homotopy types $\mathscr{M}(r_\circ, H_\circ, F_\circ)/_{\simeq}$ of manifolds in $\mathscr{M}(r_\circ, H_\circ, F_\circ)$ can now be described in the following way:

PROPOSITION 3. *The assignment* $X \mapsto (\alpha^{-1}(w_2(X)), \alpha^*[p_1(X)+24T])$ *(modulo* $\mathrm{Aut}(F_\circ)$*) defines an injection.*

$$I: \mathscr{M}(r_\circ, H_\circ, F_\circ)/_{\simeq} \to {}_{\mathrm{Aut}(F_\circ)}\backslash \bar{H}_\circ \times H_\circ^\vee/_{48H_\circ^\vee}/_{U_{F_\circ}}.$$

*Proof.* Suppose $X$ and $X'$ are manifolds in $\mathscr{M}(r_\circ, H_\circ, F_\circ)$, $\alpha: H_\circ \to H^2(X,\mathbf{Z})$ and $\alpha': H_\circ \to H^2(X',\mathbf{Z})$ isomorphisms with $\alpha^* F_X = F_\circ$ and $(\alpha')^* F_{X'} = F_\circ$. $X$ and $X'$ have the same image under $I$ iff there exists an automorphism $\gamma \in \mathrm{Aut}(F_\circ)$ with $\gamma\alpha^{-1}(w_2(X)) = (\alpha')^{-1} w_2(X')$ and $(\gamma^{-1})^*\alpha^*[p_1(X)+24T] = (\alpha')^*[p_1(X')+24T']$. Consider $\beta := \alpha \circ \gamma \circ \alpha^{-1}: H^2(X,\mathbf{Z}) \to H^2(X',\mathbf{Z})$; $\beta$ is obviously an isomorphism with $\beta^* F_{X'} = F_X$, $\beta w_2(X) = w_2(X')$, and $\beta^*[p_1(X')+24T'] = [p_1(X)+24T]$; but this means that the systems of invariants associated with $X$ and $X'$ are weakly equivalent, and therefore $X$ and $X'$ oriented homotopy equivalent.

A complete description of the set $\mathscr{M}(r_\circ, H_\circ, F_\circ)/_{\simeq}$ i.e. of the image of $I$ is only possible if the automorphism group $\mathrm{Aut}(F_\circ)$ is known; this can be a serious problem, but we will see that the 'general' automorphism group is finite (and usually small), so that the next proposition gives a reasonable estimate for the number of elements in $\mathscr{M}(r_\circ, H_\circ, F_\circ)/_{\simeq}$.

PROPOSITION 4. *Fix* $r_\circ \in \mathbf{N}$, *a finitely generated free abelian group* $H_\circ$, *and a symmetric trilinear form* $F_\circ \in S^3 H_\circ^\vee$ *which admits characteristic elements. Set* $b := rk_{\mathbf{Z}} H_\circ$, $s := rk_{\mathbf{Z}/2}(\bar{F}_\circ^t)$, *and let* $t := rk_{\mathbf{Z}/2}(\cdot_{\bar{F}_\circ})$ *be the* $\mathbf{Z}/2$*-rank of the* $\mathbf{Z}/2$*-linear square map* $\cdot_{\bar{F}_\circ}: \bar{H}_\circ \to \bar{H}_\circ^\vee$ *sending* $\bar{u} \in \bar{H}_\circ$ *to* $\bar{u}^2 \in \bar{H}_\circ^\vee$. *Then* $\mathscr{M}(r_\circ, H_\circ, F_\circ)/_{\simeq}$ *contains at most* $2^{2b-s-t}$ *elements.*

*Proof.* Fix any admissible system of invariants $(r_\circ, H_\circ, w_\circ, \tau_\circ, F_\circ, p_\circ)$ for a manifold in $\mathscr{M}(r_\circ, H_\circ, F_\circ)$. Given $(r_\circ, H_\circ, F_\circ)$, we know from the last lemma that the possible elements $w_\circ$ form a coset of $\mathrm{Ker}(\bar{F}_\circ^t)$ in $\bar{H}_\circ$, so that there exist precisely $2^{b-s}$ such elements. It remains to count the classes $[l] \in H_\circ^\vee/_{48H_\circ^\vee}/_{U_{F_\circ}}$, such that the $\mathrm{Aut}(F_\circ)$-orbit of $(w_\circ, [p_\circ + 24T_\circ + l])$ lies in the image of $I$.

To understand the latter condition we fix integral liftings $W_\circ, \in H_\circ$, $T_\circ \in H_\circ^\vee$ of $w_\circ$ and $\tau_\circ$ satisfying the admissibility conditions

i) $W_\circ^3 \equiv (p_\circ + 24T_\circ)(W_\circ) \pmod{48}$

ii) $p_\circ(x) \equiv 4x^3 + 6x^2 W_\circ + 3x W_\circ^2 \pmod{24}\ \forall x \in H_\circ$.

Clearly the $\mathrm{Aut}(F_\circ)$-orbit of $(w_\circ, [p_\circ + 24T_\circ + l])$ lies in the image of $I$ if and only if i′) $W_o^3 \equiv (p_o + 24T_o + l)(W_o) \pmod{48}$,

ii′) $(p_o + l)(x) \equiv 4x^3 + 6x^2 W_o + 3x W_o^2 \pmod{24}\ \forall x \in H_o$,

which is equivalent to $l(W_o) \equiv 0 \pmod{48}$, and $l \equiv 0 \pmod{24H_o^\vee}$ because of i) and ii).

Now, by definition of the subgroup $U_{F_o} \subset H_o^\vee/_{48H_o^\vee}$ we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & \mathrm{Ker}(\cdot_{\bar F_o}) & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 \to \mathrm{Ker}(24\cdot_{\bar F_o}) & \hookrightarrow & H_o/_{2H_o} & \xrightarrow{24\cdot\bar F_o} & U_{F_o} & \to & 0 \\
 & & \cdot_{\bar F_o}\downarrow & & \downarrow & & \\
0 & \to & H_o^\vee/_{2H_o^\vee} & \xrightarrow{24} & H_o^\vee/_{48H_o^\vee} & \to & H_o^\vee/_{24H_o^\vee} \to 0 \\
 & & \downarrow & & \downarrow & & \| \\
0 & \to & \mathrm{Coker}(\cdot_{\bar F_o}) & \to & H_o^\vee/_{48H_o^\vee}/_{U_{F_o}} & \to & H_o^\vee/_{24H_o^\vee} \to 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

The number of elements $[l] \in H_o^\vee/_{48H_o^\vee}/_{U_{F_o}}$ to be counted coincides therefore with the cardinality of the kernel of the map $ev(w_o): \mathrm{Coker}(\cdot_{\bar F_o}) \to \mathbf{Z}_{/2}$ induced by evaluation in $w_o$. This number is at most $2^{b-t}(2^{b-t-1}$ if $w_o \neq 0$ and $t \neq b$).

COROLLARY 2. *If the* $\mathbf{Z}_{/2}$*-rank* $s = rk_{\mathbf{Z}/2}(\cdot_{\bar F_o})$ *is maximal, then* $\mathcal{M}(r_o, H_o, F_o)/_{\simeq}$ *contains at most one class.*

*Proof.* Suppose $\cdot_{\bar F_o}: \bar H_o \to \bar H_o^\vee$ is surjective; then $\bar F_o^t: \bar H_o \to S^2 \bar H_o^\vee$ must have a trivial kernel, since $\bar h \bar x^2 = 0$ for all $\bar x \in \bar H_o$ implies $\bar h = 0$ if every linear form is a square. But this means $s = t = b$, so that $\mathcal{M}(r_o, H_o, F_o)/_{\simeq}$ has at most one element.

EXAMPLE 4. Let $H_o = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$, $e_1^3 = a$, $e_1^2 e_2 = b$, $e_1 e_2^2 = c$, $e_2^3 = d$. If $\bar b \equiv \bar c \pmod 2$, and $\bar a \bar d - \bar b \bar c \equiv 1 \pmod 2$, then $\mathcal{M}(r_o, H_o, F_o)/_{\simeq}$ contains precisely one class for every $r_o \geqslant 0$.