7. More on groups of type F

(Both $\bar{L}^h(f;k)$ and $L^h(f;k)$ have an interpretation in terms of Nielsen fixed point theory, but we will not make use of this.)

Theorem 6.16 together with the proof of (6.12) yields the following formula. For all sufficiently large μ:

$$\chi_1(X;\mathbf{Q})(\gamma) = \sum_{i=\mu+1}^{\mu+\nu q} (\tfrac{1}{i}\bar{L}^h(f^i;\mathbf{Q}), -L(f^i))\,.$$

Since this formula is valid for *all* sufficiently large μ, it is easy to see (because of periodicity and the appearance of the coefficients $\frac{1}{i}$) that:

COROLLARY 6.17. *For all sufficiently large* i, $\bar{L}^h(f^i;\mathbf{Q}) = 0$. □

Thus:

COROLLARY 6.18. *For all sufficiently large* μ:

$$\chi_1(X;\mathbf{Q})(\gamma) = \left(0, -\sum_{i=\mu+1}^{\mu+\nu q} L(f^i)\right).$$

In particular, if f *is also homotopy equivalence*

$$\chi_1(X;\mathbf{Q})(\gamma) = \left(0, -\sum_{i=0}^{\nu q-1} L(f^i)\right). \quad \square$$

7. MORE ON GROUPS OF TYPE $\mathscr{F}$

We consider in more detail the special case of the mapping torus of a homotopy equivalence of an aspherical complex.

Let H be an arbitrary group, let $\theta: H \to H$ be an automorphism, and let G be the semidirect product $\langle H, t \mid tht^{-1} = \theta(h) \text{ for all } h \in H\rangle$. Write $\mathrm{Fix}(\theta) = \{h \in H \mid \theta(h) = h\}$ and write $\langle x\rangle$ for the cyclic subgroup generated by $x \in G$. Let $\mathrm{Out}(H) = \mathrm{Aut}(H)/\mathrm{Inn}(H)$ be the group of outer automorphisms of H, i.e. the quotient of the group, $\mathrm{Aut}(H)$, of automorphisms of H by the normal subgroup $\mathrm{Inn}(H)$ of inner automorphisms.

LEMMA 7.1. *If* θ *has infinite order in* $\mathrm{Out}(H)$, *then* $Z(G) = Z(H) \cap \mathrm{Fix}(\theta)$. *If* θ *has finite order* r *in* $\mathrm{Out}(H)$, *and* $h_0 \in H$ *is such that* $\theta^r(\cdot) = h_0(\cdot)h_0^{-1}$, *there are two cases:*

(1) *No positive power of* h_0 *lies in* $Z(H)\mathrm{Fix}(\theta)$. *Then* $Z(G) = Z(H) \cap \mathrm{Fix}(\theta)$.

(2) *Some positive power of* h_0 *lies in* $Z(H)\operatorname{Fix}(\theta)$. *Let* p *be the smallest positive integer such that* $h_0^{-p} \in Z(H)\operatorname{Fix}(\theta)$ *and let* $x = uh_0^{-p}t^{rp}$ *where* $u \in Z(H)$ *is such that* $uh_0^{-p} \in \operatorname{Fix}(\theta)$. *Then* $Z(G) = (Z(H) \cap \operatorname{Fix}(\theta))\langle x \rangle$.

Proof. Suppose $ht^m \in Z(G)$ where $h \in H$. Then $h\theta^m(h') = h'\theta^n(h)$ for every $h' \in H$, $n \in \mathbf{Z}$. In particular, taking $h' = 1$ and $n = 1$, $h \in \operatorname{Fix}(\theta)$. Taking h' arbitrary and $n = 1$, $\theta^m(h') = h^{-1}h'h$ for all $h' \in H$. Thus, if θ has infinite order in $\operatorname{Out}(H)$ and $ht^m \in Z(G)$ then $m = 0$ and $h \in Z(H)$. So $Z(G) \subset Z(H) \cap \operatorname{Fix}(\theta)$, and the reverse inclusion is clear.

If θ has finite order r in $\operatorname{Out}(H)$ and $ht^m \in Z(G)$, the above argument shows that $m = \nu r$ for some $\nu \in \mathbf{Z}$. So $\theta^{\nu r}(\cdot) = h^{-1}(\cdot)h = h_0^{\nu}(\cdot)h_0^{-\nu}$, implying $hh_0^{\nu} \in Z(H)$. Conversely, it is straightforward to show that any $ht^{\nu r}$ with $h \in \operatorname{Fix}(\theta) \cap h_0^{-\nu}Z(H)$ lies in $Z(G)$; hence: $Z(G) = \{ht^{\nu r} \in G \mid \nu \in \mathbf{Z},\ h \in \operatorname{Fix}(\theta) \cap h_0^{-\nu}Z(H)\}$. If no positive power of h_0 lies in $Z(H)\operatorname{Fix}(\theta)$ then $ht^{\nu r} \in Z(G)$ if and only if $\nu = 0$ and $h \in Z(H) \cap \operatorname{Fix}(\theta)$. If some positive power of h_0 lies in $Z(H)\operatorname{Fix}(\theta)$, let p and u be as above. Then any $ht^{\nu r} \in Z(G)$ can be written as $(hh_0^{np}u^{-n})(uh_0^{-p}t^{rp})^n$ where $\nu = np$ (observe that $hh_0^{np}u^{-n} \in Z(H) \cap \operatorname{Fix}(\theta)$). □

ADDENDUM 7.2. *If* θ *has finite order* r *in* $\operatorname{Out}(H)$ *and* $\theta^r(\cdot) = h_0(\cdot)h_0^{-1}$ *then*

$$Z(G) = \{ht^{\nu r} \in G \mid \nu \in \mathbf{Z},\ h \in \operatorname{Fix}(\theta),\ hh_0^{\nu} \in Z(H)\}.$$

Proof. In case (1) of Lemma 7.1 this is clear, and in case (2) it is part of the last proof. □

We are very grateful to Peter Neumann for providing us with the proof of the following proposition which shows that (1) in Lemma 7.1 cannot occur.

PROPOSITION 7.3. *Let* $\theta: H \to H$ *be an automorphism whose image in* $\operatorname{Out}(H)$ *has finite order* r, *and let* $h_0 \in H$ *be such that* $\theta^r(\cdot) = h_0(\cdot)h_0^{-1}$. *Then* $h_0^r \in Z(H)\operatorname{Fix}(\theta)$.

Proof. Let $\theta^r(\cdot) = h_0(\cdot)h_0^{-1}$. Since $\theta^r\theta = \theta\theta^r$, we have $\theta(h_0) = h_0\zeta$ for some $\zeta \in Z(H)$. For $i = 0, \ldots, r-1$, let $\zeta_i = \theta^i(\zeta)$. The identity $h_0 = \theta^r(h_0)$ implies that $\zeta_0\zeta_1 \cdots \zeta_{r-1} = 1$. Define $x = h_0^r\zeta_0^{r-1}\zeta_1^{r-2} \cdots \zeta_{r-2}$. Then

$$\theta(x) = h_0^r\zeta_0^r\zeta_1^{r-1} \cdots \zeta_{r-2}^2\zeta_{r-1} = h_0^r\zeta_0^{r-1}\zeta_1^{r-2} \cdots \zeta_{r-2}$$

(the second equality uses $\zeta_{r-1} = (\zeta_0 \zeta_1 \cdots \zeta_{r-2})^{-1}$ and the fact that the group generated by h_0 and $Z(H)$ is abelian). Thus $x \in \mathrm{Fix}(\theta)$ and so $h_0^r \in Z(H)\mathrm{Fix}(\theta)$. $\square$

Remark. In [GN_3] we called an automorphism θ as in Case (2) of Lemma 7.1 *special.* In view of Proposition 7.3, we abandon this terminology here.

Remark. The hypothesis of Proposition 7.3 yields a homomorphism $\mathbf{Z}/r\mathbf{Z} \to \mathrm{Out}(H)$ and hence a homomorphism $\mathbf{Z}/q\mathbf{Z} \to \mathrm{Out}(H)$ for any multiple q of r. There is a well-known obstruction $O_q \in H^3(\mathbf{Z}/q\mathbf{Z}, Z(H))$ whose vanishing is equivalent to the existence of an extension $1 \to H \to E \to \mathbf{Z}/q\mathbf{Z} \to 1$ with the given outer action. The content of Proposition 7.3 is that $O_{r^2} = 0$. For more on this, see [GN_4].

Combining Lemma 7.1 and Proposition 7.3, we have the following structure theorem for the center of the semidirect product G:

THEOREM 7.4. *If* θ *has infinite order in* $\mathrm{Out}(H)$, *then* $Z(G) = Z(H) \cap \mathrm{Fix}(\theta)$. *If* θ *has finite order* r *in* $\mathrm{Out}(H)$ *then* $Z(G) = (Z(H) \cap \mathrm{Fix}(\theta))\langle x \rangle$ *where* $x = uh_0^{-p}t^{rp}$, p *is the smallest positive integer dividing* r *such that* $h_0^{-p} \in Z(H)\mathrm{Fix}(\theta)$ *and* $u \in Z(H)$ *is such that* $uh_0^{-p} \in \mathrm{Fix}(\theta)$. $\square$

Definition 7.5. Let $\theta: H \to H$ be an automorphism whose image in $\mathrm{Out}(H)$ has finite order r, and let $h_0 \in H$ be such that $\theta^r(\cdot) = h_0(\cdot)h_0^{-1}$. The *period* of θ is the integer $q = pr$ where p is the least positive integer such $h_0^{-p} \in Z(H)\mathrm{Fix}(\theta)$.

Note that Proposition 7.3 guarantees that the period q exists. It is straightforward to show that q depends only on the image of θ in $\mathrm{Out}(H)$. From Definition 7.5, we have that r, the order of θ in $\mathrm{Out}(H)$, divides q and by Proposition 7.3 q divides r^2.

PROPOSITION 7.6. *Suppose* $\theta: H \to H$ *has finite order* m *in* $\mathrm{Aut}(H)$. *Then the period of* θ *divides* m.

Proof. Let $h_0 \in H$ be such that $\theta^r(\cdot) = h_0(\cdot)h_0^{-1}$ where r is the order of the image of θ in $\mathrm{Out}(H)$. Then $h_0^n \in Z(H) \subset Z(H)\mathrm{Fix}(\theta)$ where $n = m/r$. $\square$

We give some sufficient conditions for the period of an automorphism to coincide with its order in $\mathrm{Out}(H)$.

PROPOSITION 7.7. *Suppose* $\theta: H \to H$ *has finite order* r *in* $\mathrm{Out}(H)$, *the restriction of* θ *to* $Z(H)$ *is the identity, and* $Z(H)$ *and has no* l*-torsion for* l *dividing* r. *Then the period of* θ *is* r.

Proof. Let $\theta^r(\cdot) = h_0(\cdot)h_0^{-1}$. Using $\theta^r\theta = \theta\theta^r$, we have $\omega \equiv h_0\theta(h_0^{-1}) \in Z(H)$. The restriction of θ to $Z(H)$ is the identity so $\omega = \theta^j(\omega) = \theta^j(h_0)\theta^{j+1}(h_0^{-1})$ for any j. Thus $\omega^r = \prod_{j=0}^{r-1}\theta^j(h_0)\theta^{j+1}(h_0^{-1}) = h_0\theta^r(h_0^{-1}) = 1$. Since $Z(H)$ has no l-torsion for l dividing r, $\omega = 1$. Hence $h_0 \in \mathrm{Fix}(\theta)$. $\square$

A similar argument shows:

PROPOSITION 7.8. *Suppose* $\theta: H \to H$ *has finite odd order* r *in* $\mathrm{Out}(H)$ *and the restriction of* θ *to* $Z(H)$ *is given by* $h \mapsto h^{-1}$. *Then the period of* θ *is* r. $\square$

Let Z be a (not necessarily finite) $K(H, 1)$ complex and let $f: Z \to Z$ be a continuous map which induces θ (after choosing a basepoint and basepath). The homomorphism $(p_f)_*: G \to \mathbf{Z}$ of §6 is identified with $ht^m \mapsto -m$. Since $\Gamma = \pi_1(\mathscr{C}(Z), \mathrm{id}) \cong Z(G)$, the rotation degree homomorphism $P_*: Z(G) \to \mathbf{Z}$ is just the restriction of $(p_f)_*$. We immediately conclude from Theorem 7.4:

COROLLARY 7.9. *There is an an exact sequence* $0 \to Z(H) \cap \mathrm{Fix}(\theta) \to Z(G) \xrightarrow{P_*} \mathbf{Z}$ *such that* $P_*(Z(G)) = q\mathbf{Z}$ *where* $q = 0$ *if* θ *has infinite order in* $\mathrm{Out}(H)$ *and* $q > 0$ *is the period of* θ *if the image of* θ *has finite order in* $\mathrm{Out}(H)$. $\square$

Theorem 6.3 and the discussion preceding it yield:

PROPOSITION 7.10. *The map* f *is an eventually coherent periodic homotopy idempotent of period* $q > 0$ *if and only if* θ *has finite order in* $\mathrm{Out}(H)$ *and has period* $q > 0$. $\square$

Now suppose H is of type $\mathscr{F}$ so that we may take Z to be a finite $K(H, 1)$ complex. Assume f is cellular. Then $X = T(Z, f)$ is a finite $K(G, 1)$ complex. By (6.12) and Proposition 6.18,

THEOREM 7.11. *If* θ *has infinite order in* $\mathrm{Out}(H)$ *then* $\chi_1(G) = 0$. *If* θ *has finite order* r *in* $\mathrm{Out}(H)$ *and period* $q > 0$ *then*

$$\chi_1(G)(ht^{\nu q}) = \left(\sum_{n \geqslant 0} \sum_{i=0}^{\nu q - 1} (-1)^n A(\operatorname{trace}([\tilde{f}_n][f_n^i])), -(q/r)\nu \sum_{i=0}^{r-1} L(f^i) \right)$$

and

$$\chi_1(G;\mathbf{Q})(ht^{\nu q}) = \left(0, -(q/r)\nu \sum_{i=0}^{r-1} L(f^i) \right) = (q/r)\nu \sum_{i=0}^{r-1} L(f^i)\{t\}$$

where $h \in \operatorname{Fix}(\theta) \cap h_0^{-\nu q/r} Z(H)$. □

Similarly, one can read off formulae for $\tilde{X}_1(G)$ from Theorem 6.14 and the rational version from Theorem 6.16.

8. Outer automorphisms of groups of type $\mathscr{F}$

In this section we apply the preceding theory to prove the following theorem which relates the algebraic topology of an automorphism $\theta: H \to H$ of a group H of type $\mathscr{F}$ such that θ has finite order in $\operatorname{Out}(H)$ to the fixed group of θ.

THEOREM 8.1. *Let H be a group of type $\mathscr{F}$ which has the Weak Bass Property over* $\mathbf{Q}$. *Suppose that $\theta: H \to H$ is an automorphism whose order in* $\operatorname{Out}(H)$ *is* $r \geqslant 1$. *If the sum of the Lefschetz numbers $\sum_{i=0}^{r-1} L(\theta^i)$ is non-zero then* $Z(H) \cap \operatorname{Fix}(\theta) = (1)$.

Before proving this we note that the quantity $\sum_{i=0}^{r-1} L(\theta^i)$ appearing above has the following interpretation:

PROPOSITION 8.2. $\sum_{i=0}^{r-1} L(\theta^i)$ *is r times the Euler characteristic of the θ-invariant part of the homology of H, i.e.,*

$$\sum_{i=0}^{r-1} L(\theta^i) = r \sum_{j \geqslant 0} (-1)^j \operatorname{rank} \ker(\mathrm{id} - \theta_j : H_j(H) \to H_j(H)) .$$

Proof. By elementary linear algebra, for any square complex matrix A with $A^r = I$ we have $\operatorname{trace}(\sum_{i=0}^{r-1} A^i) = r \dim \ker(I - A)$. The conclusion easily follows. □

Proof of Theorem 8.1. Let G be the semidirect product $G = H \times_\theta T$ where T is infinite cyclic. By Lemma 8.7, below, G also has the WBP over $\mathbf{Q}$. Applying Theorem 7.11 to G, we have that $\chi_1(G;\mathbf{Q}) \neq 0$. By