0. Introduction

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CONCERNING A REAL-VALUED CONTINUOUS FUNCTION ON THE INTERVAL WITH GRAPH OF HAUSDORFF DIMENSION 2

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ABSTRACT. A real-valued continuous nowhere-differentiable function on [0, 1] is constructed. Its graph F is proved to have the following property. If B is a Borel subset of F and if the projection of B on [0, 1] has positive Lebesgue measure, then the Hausdorff dimension of B is two.

0. INTRODUCTION

In 1903 Takagi [TAK, p. 176] gave an extremely simple construction of a nowhere differentiable real-valued continuous function on [0, 1]. Takagi's construction is

$$T(x) = \sum_{p=0}^{\infty} 2^{-p} \operatorname{dist}(2^p x, \mathbf{Z}) \tag{1}$$

where each term is a scaled version of the sawtooth function

$$\operatorname{dist}(x, \mathbf{Z}) := \inf\{ |x - y| : y \in \mathbf{Z}\} . \tag{2}$$

Later, in 1930, van der Waerden [WAE] gave a similar example, which de Rham [RHA], in 1957, improved to an example identical with Takagi's.

It follows from a proof of Mauldin and Williams [M-W, pp. 795-797] that the graph of the Takagi function has a σ-finite linear Hausdorff measure and hence is of Hausdorff dimension 1.

In 1937 Besicovitch and Ursell [B-U, p. 29] constructed for an arbitrary α, $1 < \alpha < 2$, a real-valued nowhere-differentiable function in $C[0, 1]$ with graph of Hausdorff dimension α. They too used the sawtooth function $\operatorname{dist}(x, \mathbf{Z})$ as a building block in their construction.

In this paper we construct a real valued continuous function $f(x), x \in [0, 1]$, whose graph has an optimal property with respect to Hausdorff dimension and measure.

We prove that for an arbitrary α, $1 < \alpha < 2$, $f(x)$ has the property

$\mathscr{P}(\alpha)$: Every Borel subset $B \subset \mathrm{graph}(f)$, with projection on the x-axis of positive Lebesgue measure $m(\mathrm{Proj}(B)) > 0$, has infinite α-dimensional Hausdorff measure

$$H^{\alpha}(B) = +\infty . \tag{3}$$

It is easy to see that

$$\mathscr{P}(\alpha)\ \forall \alpha < 2 \Leftrightarrow \mathscr{P}$$

where

$\mathscr{P}$: Every Borel set $B \subset \mathrm{graph}(f)$ with $m(\mathrm{Proj}(B)) > 0$ has Hausdorff dimension equal to two.

Rather than establish a general theorem valid for a class of functions we shall construct a single function with the desired property. The rationale is to provide a simple construction accompanied by a short, clear and instructive proof.

Our function is

$$f(x) = \sum_{p=0}^{\infty} 2^{-p} \operatorname{dist}(2^{2^p} x, \mathbf{Z}) . \tag{4}$$

Even though $\mathscr{P}$ is established for only a single function f, the proof contains general methods extracted as Lemma 1 and Lemma 2. It appears that Lemma 1 is well known in more general cases than ours; compare [P-U, p. 159, the beginning of the proof of their Lemma 1]. However the proof is included here for completeness and because in the present case it is particularly simple.

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PROBLEM. We believe that the following problem is unsolved.

Part 1: Construct a real valued function in $C[0, 1]$ with graph of Hausdorff dimension 1 and with property $\mathscr{P}(\alpha)$ for $\alpha = 1$.

Part 2: Determine the optimal smoothness in terms of the second difference of such a function.

Notation. The diameter of U is denoted by $|U|$ and the L^1-norm of $g \in L^1(\mathbf{R})$ by $\|g\|$. If f is a real valued function in $C[0, 1]$, we write $\tilde{f}(x)$ for $(x, f(x))$. The notation $H^{\alpha}(F)$ stands for α-dimensional Hausdorff measure of a set $F \subset \mathbf{R}^2$ and $M^{\alpha}(F)$ is the α-dimensional net measure of F

constructed by closed dyadic cubes. The graph of a real valued function $f \in C[0, 1]$ is denoted by graph (f). By a dyadic cube we mean a cube which is the Cartesian product of dyadic intervals. If Q is an arbitrary dyadic closed cube, then the band of type $\{(x, y) : (x, z) \in Q$ for some $z \in \mathbf{R}\}$ is called a dyadic band. In our construction the dyadic bands of width 2^{-2^p} play a special role. They are called bands of generation $p, p = 0, 1, 2, \dots$.

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1. A Lemma about mass distribution

By a mass distribution on a subset A of $\mathbf{R}^2$ we mean a measure μ on A such that $0 < \mu(A) < \infty$.

LEMMA 1. *Let f be a real valued measurable function defined on $[0, 1]$. Then there is a mass distribution μ on $F :=$ graph(f) such that*

1) *for any two subintervals I and I' of $[0, 1]$, with $m(I) = m(I')$,*

$$\mu(I \times \mathbf{R}) = \mu(I' \times \mathbf{R})$$

and

2) *if for two Borel sets B_1 and B_2 in $[0, 1] \times \mathbf{R}$ there exists $(x_0, y_0) \in \mathbf{R}^2$ such that*

$$B_1 \cap F + (x_0, y_0) = B_2 \cap F$$

then

$$\mu(B_1) = \mu(B_2) .$$

Proof. Let B be an arbitrary Borel set in $\mathbf{R}^2$. Define

$$\mu(B) = m(\tilde{f}^{-1}(B)) . \tag{5}$$

Then it is obvious that μ is a mass distribution on graph(f) and 1) and 2) follow from the translation invariance of the Lebesgue measure.

2. A Lemma about mass distribution and successive translations

LEMMA 2. *Let $g(y) \geqslant 0$ and $g(y) \in L^1(\mathbf{R})$. If I is a finite interval and d is a positive real number then*

$$\int_I \sum_{n=-\infty}^{\infty} g(y - nd)\, dy < \left(1 + \operatorname{int} \frac{m(I)}{d}\right) \cdot \| g \| . \tag{6}$$