

# 0. Introduction

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **41 (1995)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

## Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

## Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

## CONCERNING A REAL-VALUED CONTINUOUS FUNCTION ON THE INTERVAL WITH GRAPH OF HAUSDORFF DIMENSION 2

by Peter WINGREN

ABSTRACT. A real-valued continuous nowhere-differentiable function on  $[0, 1]$  is constructed. Its graph  $F$  is proved to have the following property. If  $B$  is a Borel subset of  $F$  and if the projection of  $B$  on  $[0, 1]$  has positive Lebesgue measure, then the Hausdorff dimension of  $B$  is two.

### 0. INTRODUCTION

In 1903 Takagi [TAK, p. 176] gave an extremely simple construction of a nowhere differentiable real-valued continuous function on  $[0, 1]$ . Takagi's construction is

$$(1) \quad T(x) = \sum_{p=0}^{\infty} 2^{-p} \text{dist}(2^p x, \mathbf{Z})$$

where each term is a scaled version of the sawtooth function

$$(2) \quad \text{dist}(x, \mathbf{Z}) := \inf \{ |x - y| : y \in \mathbf{Z} \} .$$

Later, in 1930, van der Waerden [WAE] gave a similar example, which de Rham [RHA], in 1957, improved to an example identical with Takagi's.

It follows from a proof of Mauldin and Williams [M-W, pp. 795-797] that the graph of the Takagi function has a  $\sigma$ -finite linear Hausdorff measure and hence is of Hausdorff dimension 1.

In 1937 Besicovitch and Ursell [B-U, p. 29] constructed for an arbitrary  $\alpha$ ,  $1 < \alpha < 2$ , a real-valued nowhere-differentiable function in  $C[0, 1]$  with graph of Hausdorff dimension  $\alpha$ . They too used the sawtooth function  $\text{dist}(x, \mathbf{Z})$  as a building block in their construction.

In this paper we construct a real valued continuous function  $f(x)$ ,  $x \in [0, 1]$ , whose graph has an optimal property with respect to Hausdorff dimension and measure.

We prove that for an arbitrary  $\alpha$ ,  $1 < \alpha < 2$ ,  $f(x)$  has the property

$\mathcal{P}(\alpha)$ : Every Borel subset  $B \subset \text{graph}(f)$ , with projection on the  $x$ -axis of positive Lebesgue measure  $m(\text{Proj}(B)) > 0$ , has infinite  $\alpha$ -dimensional Hausdorff measure

$$(3) \quad H^\alpha(B) = +\infty.$$

It is easy to see that

$$\mathcal{P}(\alpha) \forall \alpha < 2 \Leftrightarrow \mathcal{P}$$

where

$\mathcal{P}$ : Every Borel set  $B \subset \text{graph}(f)$  with  $m(\text{Proj}(B)) > 0$  has Hausdorff dimension equal to two.

Rather than establish a general theorem valid for a class of functions we shall construct a single function with the desired property. The rationale is to provide a simple construction accompanied by a short, clear and instructive proof.

Our function is

$$(4) \quad f(x) = \sum_{p=0}^{\infty} 2^{-p} \text{dist}(2^{2p}x, \mathbf{Z}).$$

Even though  $\mathcal{P}$  is established for only a single function  $f$ , the proof contains general methods extracted as Lemma 1 and Lemma 2. It appears that Lemma 1 is well known in more general cases than ours; compare [P-U, p. 159, the beginning of the proof of their Lemma 1]. However the proof is included here for completeness and because in the present case it is particularly simple.

The author is grateful to Professor V.P. Havin [HAV] for suggesting the investigation of fractal graphs with respect to  $\mathcal{P}(\alpha)$ ,  $\alpha = 1$ .

**PROBLEM.** We believe that the following problem is unsolved.

*Part 1:* Construct a real valued function in  $C[0, 1]$  with graph of Hausdorff dimension 1 and with property  $\mathcal{P}(\alpha)$  for  $\alpha = 1$ .

*Part 2:* Determine the optimal smoothness in terms of the second difference of such a function.

*Notation.* The diameter of  $U$  is denoted by  $|U|$  and the  $L^1$ -norm of  $g \in L^1(\mathbf{R})$  by  $\|g\|$ . If  $f$  is a real valued function in  $C[0, 1]$ , we write  $\tilde{f}(x)$  for  $(x, f(x))$ . The notation  $H^\alpha(F)$  stands for  $\alpha$ -dimensional Hausdorff measure of a set  $F \subset \mathbf{R}^2$  and  $M^\alpha(F)$  is the  $\alpha$ -dimensional net measure of  $F$

constructed by closed dyadic cubes. The graph of a real valued function  $f \in C[0, 1]$  is denoted by  $\text{graph}(f)$ . By a dyadic cube we mean a cube which is the Cartesian product of dyadic intervals. If  $Q$  is an arbitrary dyadic closed cube, then the band of type  $\{(x, y) : (x, z) \in Q \text{ for some } z \in \mathbf{R}\}$  is called a dyadic band. In our construction the dyadic bands of width  $2^{-2^p}$  play a special role. They are called bands of generation  $p$ ,  $p = 0, 1, 2, \dots$ .

*Acknowledgement.* We would like to thank the referee for helpful suggestions.

### 1. A LEMMA ABOUT MASS DISTRIBUTION

By a mass distribution on a subset  $A$  of  $\mathbf{R}^2$  we mean a measure  $\mu$  on  $A$  such that  $0 < \mu(A) < \infty$ .

LEMMA 1. *Let  $f$  be a real valued measurable function defined on  $[0, 1]$ . Then there is a mass distribution  $\mu$  on  $F := \text{graph}(f)$  such that*

1) *for any two subintervals  $I$  and  $I'$  of  $[0, 1]$ , with  $m(I) = m(I')$ ,*

$$\mu(I \times \mathbf{R}) = \mu(I' \times \mathbf{R})$$

*and*

2) *if for two Borel sets  $B_1$  and  $B_2$  in  $[0, 1] \times \mathbf{R}$  there exists  $(x_0, y_0) \in \mathbf{R}^2$  such that*

$$B_1 \cap F + (x_0, y_0) = B_2 \cap F$$

*then*

$$\mu(B_1) = \mu(B_2).$$

*Proof.* Let  $B$  be an arbitrary Borel set in  $\mathbf{R}^2$ . Define

$$(5) \quad \mu(B) = m(\tilde{f}^{-1}(B)).$$

Then it is obvious that  $\mu$  is a mass distribution on  $\text{graph}(f)$  and 1) and 2) follow from the translation invariance of the Lebesgue measure.

### 2. A LEMMA ABOUT MASS DISTRIBUTION AND SUCCESSIVE TRANSLATIONS

LEMMA 2. *Let  $g(y) \geq 0$  and  $g(y) \in L^1(\mathbf{R})$ . If  $I$  is a finite interval and  $d$  is a positive real number then*

$$(6) \quad \int_I \sum_{n=-\infty}^{\infty} g(y - nd) dy < \left(1 + \text{int} \frac{m(I)}{d}\right) \cdot \|g\|.$$