5. Other Questions

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Example. The group $G = \mathbf{Z} \times \mathbf{Z}_2 \times \mathbf{Z}_2$ acts faithfully and chaotically on $\mathbf{T}^2$ in such a way that none of the elements of G act chaotically on $\mathbf{T}^2$ in R. Devaney's sense.

Proof. First, as described above, there exist chaotic homeomorphisms of the closed disc (and hence of the closed square) which are the identity on the boundary. Let f be such a homeomorphism. Now consider $\mathbf{T}^2$ as the unit square with vertices (i, j) with $i, j \in \{0, 1\}$ and with edges identified in the usual manner; that is $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$. Now use the x and y axes to subdivide $\mathbf{T}^2$ into 4 isometric subsquares. Let F be the homeomorphism of $\mathbf{T}^2$ obtained by applying f in each of the 4 subsquares. Let $g: \mathbf{T}^2 \to \mathbf{T}^2$ be the translation $g(x, y) = (x + 1/2, y)$. Similarly, define h by $h(x, y) = (x, y + 1/2)$. Then the group $G = \mathbf{Z} \times \mathbf{Z}_2 \times \mathbf{Z}_2$ generated by F, g and h acts chaotically on $\mathbf{T}^2$. But clearly G contains no element which acts chaotically on $\mathbf{T}^2$.

5. Other Questions

In this section we present some open questions which we have been unable to resolve. The main question is the following:

QUESTION 2. *Is there a faithful chaotic action of* $\mathbf{Z} \times \mathbf{Z}$ *on the torus* $\mathbf{T}^2$ *or the sphere* $\mathbf{S}^2$*?*

This question is of interest since in order to further the study of chaotic actions, one would naturally look to actions, on low dimensional manifolds, of groups which are simple generalizations of $\mathbf{Z}$. Because of Theorem 4, the obvious place to start is in dimension 2. Now the group $\mathbf{Z} \times \mathbf{Z} (= \mathbf{Z}^2)$ acts chaotically on $\mathbf{T}^4$. But it is not clear whether $\mathbf{Z}^2$ acts chaotically and faithfully on $\mathbf{T}^2$. Notice that $SL(2, \mathbf{Z})$ has no subgroup isomorphic to $\mathbf{Z}^2$. Indeed, $PSL(2, \mathbf{Z})$ is a free product $\mathbf{Z}_2 * \mathbf{Z}_3$ (see [MKS]) and hence by Kurosh's theorem (see [LS]), it cannot have $\mathbf{Z}^2$ as a subgroup. But $PSL(2, \mathbf{Z})$ is the quotient of $SL(2, \mathbf{Z})$ by the group $\{\pm \mathrm{Id}\} \cong \mathbf{Z}_2$. So $SL(2, \mathbf{Z})$ cannot have $\mathbf{Z}^2$ as a subgroup either.

It follows from the above discussion that if $G = \mathbf{Z}^2$ acts chaotically and faithfully on $\mathbf{T}^2$, then G cannot contain a linear hyperbolic toral automorphism. Indeed, according to [AdPa], if f is a linear hyperbolic toral automorphism and if g is a homeomorphism of $\mathbf{T}^n$ which commutes with f, then g is also a linear toral automorphism. (For more on commuting diffeomorphisms of tori, see [KaSp].)

Another general question is:

QUESTION 3. *What residually finite groups have a faithful chaotic action on some smooth connected compact manifold?*

Clearly finite groups are residually finite but have no faithful chaotic actions on any connected compact manifold. On the other hand, if a group G acts faithfully and chaotically on a compact manifold, then is G necessarily countable?, finitely generated?, discrete as a subgroup of $\mathrm{Hom}(M)$?, closed as a subgroup of $\mathrm{Hom}(M)$? These properties hold for the known examples of chaotic actions constructed from the action of $SL(n, \mathbf{Z})$ on $\mathbf{T}^n$. The properties would seem unlikely to hold in general, but counterexamples have proved to be elusive. Notice that for a smooth compact manifold M, a discrete subgroup $G \leqslant \mathrm{Hom}(M)$ is necessarily countable, since $\mathrm{Hom}(M)$ is second countable. So on smooth compact manifolds one has the following implications:

$$\begin{tikzcd}
& \text{finitely generated} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \text{discrete} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{Hopfian} & & \text{countable} & & \text{closed}
\end{tikzcd}$$

Notice that there is a simple partial result: Every topological group acting continuously, faithfully and chaotically on a Hausdorff space is totally pathwise disconnected. To see this, notice that if $G \subseteq \mathrm{Hom}(M)$ acts chaotically, then the only continuous paths in G are the constant paths. Indeed, if γ_t is a path in G and if x has finite orbit under G, then as $\gamma_t(x)$ is a continuous path in M and as $\gamma_t(x)$ belongs to the (finite) orbit of x, so $\gamma_t(x)$ is independent of t. (We remark in passing that it is easy to see that every manifold admits a non-discrete totally pathwise disconnected group of homeomorphisms.)

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