3. The Lefschetz fixed point formula

3. The Lefschetz fixed point formula

Endomorphisms of elliptic complexes

A collection $T = (T_0, \dots, T_k)$ of complex linear maps $T_i : C^\infty(E_i) \to C^\infty(E_i)$ is an endomorphism of the complex (E, d) provided

$$T_{i+1} \cdot d_i = d_i \cdot T_i$$

for all i. The T_i then induce linear maps

$$T_i^* : H^i(E, d) \to H^i(E, d)\,.$$

When M is compact, $H^i(E, d)$ is finite dimensional, and we may form $\mathrm{tr}(T_i^*)$. We define the Lefschetz number $L(T)$ of the endomorphism T to be

$$L(T) = \sum_{i=0}^{k} (-1)^i \mathrm{tr}(T_i^*)\,.$$

We are interested in the so called *geometric endomorphisms*. To define these, let $f : M \to M$ be a smooth map and for $i = 0, \dots, k$, suppose that $A_i : f^*E_i \to E_i$ is a smooth bundle map. Then for each $x \in M$, we have a linear map

$$A_{i,x} : E_{i,f(x)} \to E_{i,x}$$

from the fiber of E_i over $f(x)$, which is the fiber of f^*E_i over x, to $E_{i,x}$ the fiber of E_i over x. For any $s \in C^\infty(E_i)$, we define $T_i s \in C^\infty(E_i)$ by

$$(T_i s)(x) = A_{i,x} \cdot s\big(f(x)\big)\,.$$

We assume that the A_i are chosen so that the T_i define an endomorphism of (E, d). For a geometric endomorphism associated to f, the Lefschetz number is denoted $L(f)$.

Two Examples

1) The classic example is that of a smooth map acting on the de Rham complex. Then we have

$$\begin{aligned}
(E, d) &= \text{the de Rham complex of } M \\
f &= \text{an arbitrary map.} \\
A_i &= i\text{th exterior power of the adjoint, } df^*, \\
&\ \text{of the differential } df \text{ of } f,\ \text{extended to } T^*_{\mathbf{C}} M \\
A_{i,x} &= \wedge^i df_x^* : \wedge^i T^*_{\mathbf{C}} M_{f(x)} \to \wedge^i T^*_{\mathbf{C}} M_x.
\end{aligned}$$

Then T_i is the familiar $f_i^* : C^\infty(\wedge^i T^*_{\mathbf{C}} M) \to C^\infty(\wedge^i T^*_{\mathbf{C}} M)$

2) An action of a compact Lie group G on a Spin manifold M is an action of G by orientation preserving isometries which preserves the given Spin structure on M. G then acts on E^+ and E^- and so also on $C^\infty(E^+)$ and $C^\infty(E^-)$ and this action commutes with D^+. Thus each $g \in G$ is an endomorphism of $(E^\pm, D^+)$ and $g \to L(g)$ defines a character of the group G. When $g = 1$, this is just the Spinor index of M.

Our aim is to relate the Lefschetz number of a geometric endomorphism to invariants defined on the fixed point set of f. To do so we need f to be non-degenerate along its fixed point set in the sense that at each fixed point p, $df_p : TM_p \to TM_p$ has no eigen vectors with eigenvalue $+1$ in directions *transverse* to the fixed point set. Such fixed points are called non-degenerate. Note in particular that $f = id_M$ satisfies this condition !

THEOREM 3.1 (The Lefschetz Theorem, [AB], [AS]). *Let f and (E,d) be as above and T a geometric endomorphism of (E,d) for f. Assume that M is compact and oriented and let M_f be the fixed point set of f. Then $L(f)$ is given by an integral over M_f of characteristic cohomology classes on M_f determined by local data on M_f.*

The general formula for L(f) is quite complicated. We will give the formula for the Spin case (see [AH], p. 20). Let G be as in 2. above acting on a compact Spin manifold M of dimension $n = 2\ell$. Fix $g \in G$. The normal bundle V_g of M_g in M has a canonical decomposition invariant under g,

$$V_g = \bigoplus_{\lambda} V_g(\lambda)$$

where $\lambda \in S^1 \subset \mathbf{C}$ and g acts on $V_g(\lambda)$ by multiplication by λ. Only a finite number of λ actually occur and we assume $\lambda = -1$ does not occur. (In applications it does not). Thus V_g is a complex bundle and V_g and M_g are canonically oriented. For every complex number $z \neq 1$, set

$$Q_z(x) = z^{1/2} e^{-x/2} \big/ (1 - z e^{-x}) .$$

Denote the associated multiplicative sequence by $B(\ ,z)$. Because of the factor of $z^{1/2}$, it is only defined up to sign. In [AH], p.21 it is explained how to remove this ambiguity. Then

$$L(g) = (-1)^\ell \int_{M_g} \widehat{\mathcal{A}}(TM_g) \cdot \prod_{\lambda} B\big(V_g(\lambda), \lambda\big) .$$

Outline of the proof of the Lefschetz Theorem

We outline a proof which does not rely in an essential way on the compactness of M. This allows us to generalize these results to complexes and endomorphisms defined along the leaves of a foliation of a compact manifold even though the leaves may be non-compact. A general reference for the material in this section is [RS].

We begin by redefining $e^{-t\Delta_i}$. Let C be the curve in the complex plane

$$C = \{z = (x, y) \mid y^2 = x + 1\}$$

and set

$$e^{-t\Delta_i} = \frac{1}{2\pi i} \int_C \frac{e^{-t\lambda}}{(\lambda I - \Delta_i)} d\lambda ,$$

i.e.

$$(e^{-t\Delta_i} s)(x) = \frac{1}{2\pi i} \int_C e^{-t\lambda} \left[(\lambda I - \Delta_i)^{-1} s \right] (x)\, d\lambda$$

for $s \in L^2(E_i)$. A Riemannian manifold has bounded geometry if its curvature is bounded and its injectivity radius is bounded away from zero. On any complete manifold of bounded geometry, $(\lambda I - \Delta_i)^{-1}$ is a bounded operator on $L^2(E_i)$ for all $\lambda \in C$ so $e^{-t\Delta_i}$ is defined.

Note that when M is compact, this agrees with our previous definition. To see this, use Cauchy's Theorem to show that the two definitions agree on an orthonormal basis.

Some facts about $e^{-t\Delta_i}$

Assume that M is a complete manifold of bounded geometry. Then

1. As before, $e^{-t\Delta_i}$ is a smoothing operator with smooth Schwartz kernel $k_t^i(x, y)$ (ref. [S]), so if M is compact, it is of trace class.
2. $\pi_{\ker \Delta_i}$, the projection onto the kernel of Δ_i, is a smoothing operator, so if M is compact, it is of trace class.
3. $\lim_{t \to \infty} e^{-t\Delta_i} = \pi_{\ker \Delta_i}$ in the strong operator topology, so if M is compact, it follows that
$$\lim_{t\to\infty} \operatorname{tr}(e^{-t\Delta_i}) = \operatorname{tr}(\pi_{\ker \Delta_i})$$
4. Let T_i be as in the Lefschetz Theorem. Then $T_i e^{-t\Delta_i}$ is a smoothing operator with Schwartz kernel
$$k_t^{T_i}(x, y) = A_{i,x} k_t^i(f(x), y) ,$$

and if M is compact it has trace

$$\operatorname{tr}(T_i \cdot e^{-t\Delta_i}) = \int_M \operatorname{tr}\big(k_t^{T_i}(x,x)\big)\,dx\,.$$

5. As $t \to 0$, if $x \neq y$, $k_t^i(x,y) \to 0$ to infinite order and this convergence is uniform in distance (x,y). Roughly speaking, this is because as $t \to 0$, $e^{-t\Delta_i} \to I$, and its Schwartz kernel is converging to the distribution on $M \times M$ which on each $\{x\} \times M$ is the Dirac δ distribution at x. Thus if $f(x) \neq x$, we have

$$\lim_{t\to 0} \operatorname{tr}\big(k_t^{T_i}(x,x)\big) = \lim_{t\to 0} \operatorname{tr}\big(A_{i,x}k_t^i\big(f(x),x\big)\big) = 0\,,$$

and given $\varepsilon > 0$, this convergence is uniform for all x with distance $\big(x,f(x)\big) > \varepsilon$.

Now suppose that M is compact and consider

$$A(t) = \sum_{i=0}^{k}(-1)^i \operatorname{tr}(T_i \cdot e^{-t\Delta_i}) = \sum_{i=0}^{k}(-1)^i \int_M \operatorname{tr}\big(k_t^{T_i}(x,x)\big)\,dx\,.$$

By 5. above, $\lim\limits_{t\to 0} A(t)$ can be computed by integrating only over a neighborhood of the fixed point set M_f of f. This integration can be done using only *local* information about (E,d), f and T_i on M_f. Thus $\lim\limits_{t\to 0} A(t)$ gives the right hand side of the Lefschetz Theorem.

6. By 3. above and the fact that $\ker\Delta_i \simeq H^i(E,d)$ we have

$$\begin{aligned}
\lim_{t\to\infty} \operatorname{tr}(T_i e^{-t\Delta_i}) &= \operatorname{tr}(T_i \cdot \pi_{\ker\Delta_i}) \\
&= \operatorname{tr}(T_i \cdot \pi^2_{\ker\Delta_i}) \\
&= \operatorname{tr}(\pi_{\ker\Delta_i} \cdot T_i \cdot \pi_{\ker\Delta_i}) \\
&= \operatorname{tr}(T_i^*)\,.
\end{aligned}$$

Then $\lim\limits_{t\to\infty} A(t)$ gives the left hand side of the Lefschetz Theorem, so to complete the proof of the Lefschetz Theorem we need only show :

THEOREM 3.2. *$A(t) = \sum_{i=0}^{k}(-1)^i \operatorname{tr}\,(T_i \cdot e^{-t\Delta_i})$ is independent of t.*

Proof. Set

$$\phi(\Delta_i) = e^{-t_1\Delta_i} - e^{-t_2\Delta_i} = \Delta_i\psi(\Delta_i)$$

where

$$\psi(x) = \frac{e^{-t_1 x} - e^{-t_2 x}}{x}.$$

Now formally we have

$$\sum_{i=0}^{k}(-1)^i \operatorname{tr}(T_i e^{-t_1\Delta_i}) - \sum_{i=0}^{k}(-1)^i \operatorname{tr}(T_i e^{-t_2\Delta_i})$$
$$= \sum_{i=0}^{k}(-1)^i \operatorname{tr}\big(T_i \phi(\Delta_i)\big)$$
$$= \sum_{i=0}^{k}(-1)^i \operatorname{tr}\big(T_i \Delta_i \psi(\Delta_i)\big)$$
$$= \sum_{i=1}^{k}(-1)^i \operatorname{tr}\big(T_i d_{i-1} d_i^* \psi(\Delta_i)\big) + \sum_{i=0}^{k-1}(-1)^i \operatorname{tr}\big(T_i d_{i+1}^* d_i \psi(\Delta_i)\big)$$

We now show that the first sum is the negative of the second.

$$\sum_{i=1}^{k}(-1)^i \operatorname{tr}\big(T_i d_{i-1} d_i^* \psi(\Delta_i)\big)$$
$$= \sum_{i=1}^{k}(-1)^i \operatorname{tr}\big(d_{i-1} T_{i-1} d_i^* \psi(\Delta_i)\big)$$
$$= \sum_{i=1}^{k}(-1)^i \operatorname{tr}\big(T_{i-1} d_i^* \psi(\Delta_i) d_{i-1}\big)$$
$$= \sum_{i=1}^{k}(-1)^i \operatorname{tr}(T_{i-1} d_i^* d_{i-1} \psi(\Delta_{i-1})\big)$$
$$= \sum_{i=0}^{k-1}(-1)^{i+1} \operatorname{tr}(T_i d_{i+1}^* d_i \psi(\Delta_i)\big)$$

and done. Of course this manipulation is purely formal and must be justified as we are working with operators on infinite dimensional spaces and not on finite dimensional ones. For this, see [ABP] and [HL 1]. Note also that in order to have a Lefschetz Theorem for complete manifolds of bounded geometry, it is only necessary to find an appropriate trace for which the above results hold.