4. The Lefschetz Theorem for foliated manifolds

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4. The Lefschetz Theorem for foliated manifolds

Let M be a compact m dimensional manifold and F a dimension n foliation on M. Then F is an n dimensional subbundle of TM such that for any two sections $X, Y \in C^\infty(F)$, $[X, Y] \in C^\infty(F)$. The Frobenius Theorem says that for each $x \in M$, there is a neighborhood U of x and a diffeomorphism

$$\phi : \mathbf{R}^n \times \mathbf{R}^q \to U \qquad n + q = m$$

so that for all $z \in \mathbf{R}^n \times \mathbf{R}^q$.

$$d\phi(T\mathbf{R}^n_z) = F_{\phi(z)}\,.$$

Such a (U, ϕ) is called a foliation chart. Given $x \in \mathbf{R}^q$, the submanifold $\phi\big(\mathbf{R}^n \times \{x\}\big)$ is called a plaque, and is denoted P^U_x. It is a local integral submanifold of F. The submanifold $\phi\big(\{0\} \times \mathbf{R}^q\big)$ is denoted $\mathbf{R}^q_U$ and is called the transverse submanifold of (U, ϕ).

A leaf L of F is a maximal integral (i.e. $TL_x = F_x$ for all $x \in L$) submanifold of M. Thus $\dim L = n$. The Frobenius Theorem implies that through each point x in M, there passes a unique leaf, denoted L_x. Each leaf is a complete manifold of bounded geometry and the bounds are uniform for all leaves.

We now extend the Lefschetz Theorem for compact manifolds to a Lefschetz Theorem for foliations of a compact manifold. This is joint work with Connor Lazarov [HL 1]. In fact, we show how to improve the results of [HL 1] by removing the assumption that F admits a transverse invariant metric. For a K-theory version of this result, see the thesis of M-T. Benameur, [Be].

Choose a smooth metric on M. This induces a smooth metric on each leaf L, and L is complete with respect to this metric. Two different metrics on M induce quasi-isometric metrics on L.

Haefliger forms

Let $\{U_i\}$ be a finite cover of M by foliation charts. For $x \in U_i$, denote its plaque by P^i_x. If $U_i \cap U_j \neq \varnothing$ we define a local diffeomorphism f_{ij} from $\mathbf{R}^q_{U_i}$ (hereafter denoted $\mathbf{R}^q_i$) to $\mathbf{R}^q_j$ as follows:

$$f_{ij}(x) = y \text{ if and only if } P^i_x \cap P^j_y \neq \varnothing\,.$$

The f_{ij} generate the holonomy pseudogroup, denoted H, which acts on the transversal space $T = \cup_i \mathbf{R}^q_i$. We may (and do) assume that the $\mathbf{R}^q_i$ are disjoint.

Recall the following construction due to Haefliger [Ha]. Let $\Omega_c^k(T)$ be the space of bounded measurable complex valued k forms on T with compact support. Denote by $\Omega_c^k(T/H)$ the quotient of $\Omega_c^k(T)$ by the vector subspace generated by elements of the form $\alpha - h^*\alpha$ where $h \in H$ and $\alpha \in \Omega_c^k(T)$ has support contained in the range of h. Give $\Omega_c^k(T/H)$ the quotient topology of the usual sup norm topology on $\Omega_c^k(T)$. Note that $\Omega_c^k(T/H)$ does not depend of the choice of cover used to define it.

Denote by $\Omega^{p+k}(M)$ the space of bounded measurable complex valued $p+k$ forms on M. As the bundle TF is oriented, there is a continuous open surjective linear map,

$$\int_F : \Omega^{p+k}(M) \to \Omega_c^k(T/H) .$$

It is given as follows. Let $\omega \in \Omega^{p+k}(M)$ and let $\{\psi_i\}$ be a partition of unity subordinate to the cover $\{U_i\}$. Set $\omega_i = \psi_i\omega$. We may integrate ω_i along the fibers of the submersion $\pi_i : U_i \to \mathbf{R}_i^q$ to obtain $\overline{\omega}_i \in \Omega_c^k(\mathbf{R}_i^q)$. Define $\int_F \omega$ to be the class of $\Sigma\overline{\omega}_i$ in $\Omega_c^k(T/H)$. It is independent of the choices made in defining it.

DIFFERENTIAL COMPLEXES ON M ELLIPTIC ALONG F

A differential complex on M along F consists of:

a) a finite collection of finite dimensional complex vector bundles $E_0, \dots, E_k$ over M

b) a collection of smooth differential operators

$$d_i : C^\infty(E_i) \to C^\infty(E_{i+1})$$

with $d_{i+1} \cdot d_i = 0$

c) each d_i differentiates only in leaf directions.

For the sake of simplicity we assume that each d_i is first order.

Each of the classical complexes mentioned above (de Rham, Dolbeault, Signature and Twisted Spin) gives a leafwise complex on M provided that the leaves have the required structures and that these structures are coherent from leaf to leaf (i.e. come from a global structure on M). For example, in the twisted Spin case, we require that the Spin structure on the leaves comes from a principal Spin(n) bundle P over M with $P \times_{\mathrm{Spin}(n)} \mathbf{R}^n \simeq TF$, and that the leafwise auxiliary twisting bundle come from a bundle over M.

For a fixed leaf L, denote $E_i\big|_L$ by E_i^L and by $C_0^\infty(E_i^L)$ the space of smooth sections of E_i^L with compact support. The operator d_i induces one, denoted also by d_i,

$$d_i : C_0^\infty(E_i^L) \to C_0^\infty(E_{i+1}^L)$$

and on L we have the complex

$$0 \to C_0^\infty(E_0^L) \xrightarrow{d_0} C_0^\infty(E_1^L) \xrightarrow{d_1} \cdots \xrightarrow{d_{k-1}} C_0^\infty(E_k^L) \to 0\,.$$

We say that the complex (E,d) is elliptic along F provided that for each leaf L, the above complex is elliptic. We assume that (E,d) is elliptic along F.

L^2 COHOMOLOGY OF (E,d)

Choose a smooth Hermitian metric on each bundle E_i *over M*. These induce metrics on each E_i^L and these metrics are unique up to quasi-isometry. Using these metrics we construct $d_i^* : C_0^\infty(E_{i+1}^L) \to C_0^\infty(E_i^L)$ just as we did before. We then construct

$$\Delta_i^L : C_0^\infty(E_i^L) \to C_0^\infty(E_i^L)$$

and we extend Δ_i to

$$\Delta_i^L : L^2(E_i^L) \to L^2(E_i^L)$$

just as before.

DEFINITION 4.1. *The ith L^2 cohomology of (E,d) along the leaf L, denoted $H_L^i(E,d)$ is*

$$H_L^i(E,d) = \ker \Delta_i^L\,.$$

The ith L^2 cohomology of (E,d) is denoted $H^i(E,d)$ and it assigns to each leaf L the ith cohomology of (E,d) along $L, H_L^i(E,d)$.

SOME FACTS

1. $H_L^i(E,d)$ consists of smooth sections and $\dim_{\mathbf{C}} H_L^i(E,d)$ may be infinite but is always countable.
2. π_L^i, the projection of $L^2(E_i^L)$ onto $H_L^i(E,d)$, is a smoothing operator (on L) with smooth Schwartz kernel $k_L^i(x,y)$.
3. $k_L^i(x,y)$ is measurable as a function of L and bounded independently of L. In particular, tr $k_L^i(x,x)$ is a bounded measurable function on M whose restriction to each leaf L is smooth.
4. Because of 3. above, we may define the dimension of $H^i(E,d)$ to be the zero dimensional Haefliger form

$$\dim\big(H^i(E,d)\big) = \int_F \operatorname{tr}\big(k_L^i(x,x)\big)\,dx\,,$$

where for any leaf L we denote the volume form obtained from the metric on L by dx. We may also define the Euler class of (E,d) as

$$\chi(E,d) = \sum_{i=0}^{k}(-1)^i \dim H^i(E,d)\,.$$

GEOMETRIC ENDOMORPHISMS

Let $f : M \to M$ be a diffeomorphism and assume that for each leaf L of F, $f(L) \subset L$. For each i, let

$$A_i : f^*E_i \to E_i$$

be a smooth bundle map. We assume that $T_i : C^\infty(E_i) \to C^\infty(E_i)$ where $(T_i s)(x) = A_{i,x} s\big(f(x)\big)$ satisfies

$$T_i d_{i-1} = d_{i-1} T_{i-1}\,.$$

The T_i then induce maps

$$T_i^L : C_0^\infty(E_i^L) \to C_0^\infty(E_i^L)$$

satisfying

$$T_i^L d_{i-1} = d_{i-1} T_{i-1}^L\,.$$

We call such a family $T = (T_0, \ldots, T_k)$ the geometric endomorphism of (E,d) defined by f and $A = (A_0, \ldots, A_k)$. The T_i^L extend to uniformly bounded linear maps

$$T_i^L : L^2(E_i^L) \to L^2(E_i^L)\,.$$

LEFSCHETZ NUMBER OF A GEOMETRIC ENDOMORPHISM

Set $T_{i,L}^* = \pi_i^L \cdot T_i^L \cdot \pi_i^L$ and denote its Schwartz kernel by $k_L^{T_i^*}(x,y)$. Then $k_L^{T_i^*}(x,y)$ is globally bounded, smooth on $L \times L$, and measurable. Thus $\operatorname{tr}\big(k_L^{T_i^*}(x,x)\big)$ is a bounded measurable function on M which is smooth on each leaf L. We define the Lefschetz class of the geometric endomorphism T to be the Haefliger zero form

$$L(T) = \sum_{i=0}^{k}(-1)^i \int_F \operatorname{tr}\big(k_L^{T_i^*}(x,x)\big)\,dx\,.$$

For our Lefschetz Theorem we shall also need two restrictions on the fixed point set, N of f. We require:

1. $N = \bigcup_\alpha N_\alpha$ is a finite disjoint union of closed, connected submanifolds N_α, each transverse to F.
2. for each $x \in N \cap L = \bigcup_\alpha N_\alpha^L$ where $N_\alpha^L = N_\alpha \cap L$, df_x has no eigen vector (in TL_x) with eigenvalue $+1$ in directions transverse (in L!) to N_α^L.

Note in particular that $f = id_M$ satisfies these conditions.

Fixed point indices

Let $\{U_i\}$ and $\{\psi_i\}$ be as above. Suppose that for each L and α we are given a differential form a_α^L defined on N_α^L. We define the Haefliger form $\int_N a$ as

$$\int_N a = \sum_i \sum_{N_\alpha^L \cap P_x^i \neq \phi} \int_{N_\alpha^L \cap P_x^i} \psi_i a_\alpha^L .$$

Note that for any plaque P_x^i, only a finite number of N_α^L satisfy $N_\alpha^L \cap P_x^i \neq \phi$. As $\int_{N_\alpha^L \cap P_x^i} \psi_i a_\alpha^L$ is a differential form on the transversal $\mathbf{R}_i^q$ of U_i, we may also consider it as a Haefliger form for F. As above, it is not difficult to show that the Haefliger form $\int_N a$ does not depend on the choices made in defining it.

THEOREM 4.2 (The Lefschetz Theorem for Foliations [HL 1]). *Let M, F, f, T, A and (E, d) be as above. To each $N_\alpha^L \subset N$ we may associate a differential form a_α^L which depends only on local data on N_α^L so that*

$$L(T) = \int_N a .$$

The proof follows the outline given above for the classical case, done leafwise. There are some very formidable technical obstacles, but these can be overcome (see [HL 1]).

If (E, d) is the de Rham, Dolbeault, Signature or Twisted Spin complex of F, and $f = id_M$, and $T = id$, then a_j^L is the usual local integrand formula (computed on each leaf, not on M) given by the Atiyah-Singer Index Theorem. We thus have an index theorem for foliated manifolds for these operators.

(Note that Connes has also proven an index theorem for foliated manifolds, (see [C]). As he works on the holonomy coverings of the leaves of F, his theorem is related to ours as the L^2 covering index theorem is related to the ordinary index theorem.) If we take the codimension 0 foliation of M which has one leaf (namely M), we recover the Atiyah-Singer Index Theorem for these operators. In general, i.e. $f \neq id_M$, $T = f^*$, a_j^L is the usual local integrand (computed on the fixed point set in each leaf, not in M) given by the Atiyah-Singer G Index Theorem. If we take the codimension 0 foliation, we recover the Atiyah-Singer G Index Theorem and the Atiyah-Bott Lefschetz Theorem for these operators.

5. Group Actions and the Lefschetz Theorem

Let F be an oriented $2k$ dimensional foliation of a compact, oriented, Riemannian manifold M. Assume that F admits a $\mathrm{Spin}(2k)$ structure. That is, there is a principal $\mathrm{Spin}(2k)$ bundle P over M and an isomorphism of oriented bundles

$$P \times_{\mathrm{Spin}(2k)} \mathbf{R}^{2k} \simeq TF\,.$$

We may then construct the bundles $E^{\pm} = P \times_{\mathrm{Spin}(2k)} \Delta^{\pm}$. The leafwise Dirac operator D^+ is constructed using the Riemannian structure on the leaves of F which is induced from M.

Let G be a compact, connected Lie group acting by isometries on M, taking each leaf of F to itself. G then acts on TF. We assume that G also acts on P (commuting with the action of $\mathrm{Spin}(2k)$) so that the induced action on $P \times_{\mathrm{Spin}(2k)} \mathbf{R}^{2k} \simeq TF$ is the given action on TF. G then acts on the bundles $E^{\pm}$ and it commutes with the operator D^+, i.e. G is a group of geometric endomorphisms of the complex $(E^{\pm}, D^+)$.

Recall the $\widehat{\mathcal{A}}$ genus defined in Section 1.

Definition 5.1. *The $\widehat{\mathcal{A}}$ genus of F is the Haefliger zero form*

$$\widehat{\mathcal{A}}(F) = \int_F \widehat{\mathcal{A}}_{k/2}(TF)\,.$$

In particular, if k is odd, $\widehat{\mathcal{A}}(F) = 0$.

Note that we have defined $\widehat{\mathcal{A}}(F)$ as the *zero th* order part of $\int_F \widehat{\mathcal{A}}(TF)$. For an interpretation of the higher order terms of $\int_F \widehat{\mathcal{A}}(TF)$, see [He].