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(Note that Connes has also proven an index theorem for foliated manifolds, (see [C]). As he works on the holonomy coverings of the leaves of F, his theorem is related to ours as the  $L^2$  covering index theorem is related to the ordinary index theorem.) If we take the codimension 0 foliation of M which has one leaf (namely M), we recover the Atiyah-Singer Index Theorem for these operators. In general, i.e.  $f \neq id_M$ ,  $T = f^*$ ,  $a_j^L$  is the usual local integrand (computed on the fixed point set in each leaf, not in M) given by the Atiyah-Singer G Index Theorem. If we take the codimension 0 foliation, we recover the Atiyah-Singer G Index Theorem and the Atiyah-Bott Lefschetz Theorem for these operators.

## 5. GROUP ACTIONS AND THE LEFSCHETZ THEOREM

Let F be an oriented 2k dimensional foliation of a compact, oriented, Riemannian manifold M. Assume that F admits a Spin(2k) structure. That is, there is a principal Spin(2k) bundle P over M and an isomorphism of oriented bundles

$$P \times_{\operatorname{Spin}(2k)} \mathbf{R}^{2k} \simeq TF$$
.

We may then construct the bundles  $E^{\pm} = P \times_{\text{Spin}(2k)} \Delta^{\pm}$ . The leafwise Dirac operator  $D^{+}$  is constructed using the Riemannian structure on the leaves of F which is induced from M.

Let G be a compact, connected Lie group acting by isometries on M, taking each leaf of F to itself. G then acts on TF. We assume that G also acts on P (commuting with the action of Spin(2k)) so that the induced action on  $P \times_{\text{Spin}(2k)} \mathbb{R}^{2k} \simeq TF$  is the given action on TF. G then acts on the bundles  $E^{\pm}$  and it commutes with the operator  $D^+$ , i.e. G is a group of geometric endomorphisms of the complex  $(E^{\pm}, D^+)$ .

Recall the  $\widehat{\mathcal{A}}$  genus defined in Section 1.

DEFINITION 5.1. The 
$$\widehat{\mathcal{A}}$$
 genus of  $F$  is the Haefliger zero form  
 $\widehat{\mathcal{A}}(F) = \int_{F} \widehat{\mathcal{A}}_{k/2}(TF)$ .

In particular, if k is odd,  $\widehat{\mathcal{A}}(F) = 0$ .

Note that we have defined  $\widehat{\mathcal{A}}(F)$  as the zero th order part of  $\int_{F} \widehat{\mathcal{A}}(TF)$ . For an interpretation of the higher order terms of  $\int_{F} \widehat{\mathcal{A}}(TF)$ , see [He]. The Lefschetz Theorem for Foliations applied to the case  $f = id_M$ , T = id says that  $\widehat{\mathcal{A}}(F)$  is equal to the index of the leafwise Spin complex, which is just L(I). The Connes Index Theorem [C] says that it is also equal to the index of the holonomy covering leafwise Spin complex.

We now prove the theorem of the introduction, namely

THEOREM 5.2 ([HL2]). Let F be an oriented foliation of a compact oriented manifold M and assume that F admits a Spin structure. If a compact connected Lie group acts non-trivially on M as a group of isometries taking each leaf of F to itself and preserving the Spin structure on F, then the  $\widehat{A}$ genus of F is zero.

As a corollary, we have the well known result of Atiyah and Hirzebruch.

THEOREM 5.3 ([AH]). Let M be a compact connected oriented manifold which admits a Spin structure. If a compact connected Lie group acts nontrivially on M, then  $\widehat{\mathcal{A}}(M) = \int_{M} \widehat{\mathcal{A}}(TM)$  is zero.

Of course, this theorem and its proof were the inspiration for Theorem 5.2. Now let G be a compact connected Lie group acting on M by isometries taking each leaf of F to itself and preserving the Spin structure on F. We quote two results from [HL2] and refer the reader to that paper for the proofs. Note that in [HL1] and [HL2], we assume that F admits a transverse invariant measure. A careful reading of those papers shows that in fact we may disregard the invariant transverse measure and consider the traces used as taking values in the Haefliger zero forms of F and all the results remain valid. See the remarks on this in [HL3].

LEMMA 5.4. The fixed point set of the action of G is a closed submanifold of M which is transverse to F.

THEOREM 5.5. The Lefschetz number L(g) is a continuous function on G.

Proof of Theorem 5.2. We may assume  $G = S^1 \subset \mathbb{C}$ . Let N be the fixed point set of G,  $N_{\alpha}$  a connected component of N, L a leaf of F and  $y \in N_{\alpha} \cap L$ . The normal bundle to  $N_{\alpha} \cap L$  in L at y can be written as  $\oplus V_y^j$ , where  $V_y^j$  is a complex vector space and  $z \in G$  acts on  $V_y^j$  by multiplication

by  $z^{m_j}$  for some positive integer  $m_j$ . It follows that the  $V^j$  are complex G vector bundles on  $N_{\alpha} \cap L$ .

Now let  $z \in \mathbb{C}$ ,  $z \neq 1$  and consider the function  $R(x, z) = 1/(1 - ze^{-x})$ . It can be written as a formal power series in x whose coefficients are rational functions in z having a pole only at z = 1, and no pole at  $z = \infty$ . To see this, write

$$\frac{1}{1-ze^{-x}} = \sum_{k=0}^{\infty} (ze^{-x})^k = \sum_{k=0}^{\infty} z^k e^{-kx} = (1+z+z^2+z^3+\cdots)$$
$$-(z+2z^2+3z^3+\cdots)x$$
$$+(z+2^2z^2+3^2z^3+\cdots)x^2/2!$$
$$-\cdots$$

Set  $f_0(z) = 1 + z + z^2 + \cdots = 1/(1 - z)$ , and for  $n \ge 1$ , set  $f_n(z) = \sum_{k=1}^{\infty} k^n z^k$ . Then  $(-1)^n f_n(z)/n!$  is the coefficient of  $x^n$  in R(x, z) and it is obvious that  $f_{n+1}(z) = zf'_n(z)$ . An induction argument then shows that  $f_n(z)$  is a rational function of z with a pole only at z = 1 and no pole at  $z = \infty$ . By induction we also have that  $z^{1/2}f_n(z)$  has a pole only at z = 1 and, as it is  $\mathcal{O}(z^{-1/2})$  at  $z = \infty$ , it has no pole at  $z = \infty$ .

Now for fixed  $z \neq 1$ , set  $Q(x,z) = z^{1/2}e^{-x/2}R(x,z)$ , which is a formal power series in x. Denote the corresponding multiplicative sequence by  $B(,z) = (B_0(,z), B_1(,z), \ldots)$ .

Let  $z \in G = S^1$  be a topological generator (i.e. z generates a dense subgroup). Then the fixed point set of z is N and z acts on  $V^j$  by multiplication by  $z^{m_j}$ . Let  $d_j$  be the complex dimension of  $V^j$  and set

$$B(V^j, z) = B_{d_i}(V^j, z^{m_j}).$$

 $B(V^j, z)$  is a cohomology class on  $N_\alpha \cap L$  whose coefficients are rational functions of z having poles only at roots of unity and no pole at  $z = \infty$ . Set

$$B(N_{lpha}\cap L,z)=\prod_{j}B(V^{j},z)\,.$$

As  $B(V^j, z)$  contains the factor  $(z^{m_j d_j})^{1/2}$ ,  $B(N_\alpha \cap L)$  contains the factor  $(z^d)^{1/2}$ ,  $d = \sum m_j d_j$ , and so is defined only up to sign. The choice of sign is determined as in [AH], page 21.

The Riemannian connection on TM over  $N_{\alpha} \cap L$  preserves the bundles  $V^{j}$  and is a complex connection on each  $V^{j}$ . Using this connection and the Riemannian connection on  $T(N_{\alpha} \cap L)$ , we may construct the differential form

 $w_{\alpha}^{L}(z)$  on  $N_{\alpha} \cap L$  which represents the cohomology class  $\widehat{\mathcal{A}}(N_{\alpha} \cap L)B(N_{\alpha} \cap L, z)$ . Then  $w_{\alpha}^{L}(z)$  is the form  $a_{\alpha}^{L}$  given in the foliation Lefschetz theorem for z acting on the leafwise Spin complex, and it defines a smooth form  $w_{\alpha}(z)$  on  $N_{\alpha}$ . Thus for  $z \in S^{1}$ , z not a root of unity, we have

$$L(z) = \int_{N} w(z) = \sum_{\alpha} \int_{N_{\alpha}} w_{\alpha}(z) \,.$$

Now notice that the right side of this equation defines a function A(F, z) on the complex plane with values in the Haefliger forms of F. Also note that A(F, z) has poles only at roots of unity and no pole at  $z = \infty$ , since  $w_{\alpha}(z)$ has poles only at roots of unity and no pole at  $z = \infty$ . Because of the factor of  $(z^d)^{1/2}$ , A(F, 0) = 0. For  $z \in S^1$ , z not a root of unity, A(F, z) = L(z). But L(z) is defined for all  $z \in S^1$  and by Theorem 5.5 it is continuous on  $S^1$ . Thus A(F, z) has no poles at all. Since it is analytic and bounded, it is constant and hence is identically zero. Therefore L(z) = 0 for all  $z \in S^1$ , but  $L(1) = \widehat{A}(F)$  so we are done.

The compactness of G is essential, as in [HL2], we give an example of an infinite discrete group acting by leaf preserving isometries on a compact oriented foliated manifold M, F and G preserves a Spin structure on F. The foliation F admits an invariant transverse measure which defines a map from the Haefliger zero forms of F to C. The image of  $\widehat{\mathcal{A}}(F)$  under this map is non-zero, so  $\widehat{\mathcal{A}}(F) \neq 0$ .

# 6. THE RIGIDITY THEOREM OF WITTEN

In 1986, Witten [W] predicted rigidity theorems for the indices of certain elliptic operators on manifolds with  $S^1$  actions. The genesis for Witten's conjecture was his study of the Dirac operator on the free loop space  $\mathcal{L}M$  (an infinite dimensional manifold) of a Spin manifold M.  $\mathcal{L}M$  admits a natural  $S^1$ action whose fixed point set is diffeomorphic to M. The sequences of bundles R(q) and R'(q) described below were derived from the normal bundle of Min  $\mathcal{L}M$  and from the formal analogue on  $\mathcal{L}M$  of the fixed point formula for the Dirac operator in the finite dimensional case.

Let  $D: C^{\infty}(E_1) \to C^{\infty}(E_2)$  be an elliptic operator on a compact manifold M and suppose M admits an  $S^1$  action preserving D. Then as noted above, Index(D) is a virtual  $S^1$  module and has a decomposition into a finite sum of irreducible complex one dimensional representations