5. Group Actions and the Lefschetz Theorem

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(Note that Connes has also proven an index theorem for foliated manifolds, (see [C]). As he works on the holonomy coverings of the leaves of F, his theorem is related to ours as the L^2 covering index theorem is related to the ordinary index theorem.) If we take the codimension 0 foliation of M which has one leaf (namely M), we recover the Atiyah-Singer Index Theorem for these operators. In general, i.e. $f \neq id_M$, $T = f^*$, a_j^L is the usual local integrand (computed on the fixed point set in each leaf, not in M) given by the Atiyah-Singer G Index Theorem. If we take the codimension 0 foliation, we recover the Atiyah-Singer G Index Theorem and the Atiyah-Bott Lefschetz Theorem for these operators.

5. Group Actions and the Lefschetz Theorem

Let F be an oriented $2k$ dimensional foliation of a compact, oriented, Riemannian manifold M. Assume that F admits a $\mathrm{Spin}(2k)$ structure. That is, there is a principal $\mathrm{Spin}(2k)$ bundle P over M and an isomorphism of oriented bundles

$$P \times_{\mathrm{Spin}(2k)} \mathbf{R}^{2k} \simeq TF .$$

We may then construct the bundles $E^{\pm} = P \times_{\mathrm{Spin}(2k)} \Delta^{\pm}$. The leafwise Dirac operator D^+ is constructed using the Riemannian structure on the leaves of F which is induced from M.

Let G be a compact, connected Lie group acting by isometries on M, taking each leaf of F to itself. G then acts on TF. We assume that G also acts on P (commuting with the action of $\mathrm{Spin}(2k)$) so that the induced action on $P \times_{\mathrm{Spin}(2k)} \mathbf{R}^{2k} \simeq TF$ is the given action on TF. G then acts on the bundles $E^{\pm}$ and it commutes with the operator D^+, i.e. G is a group of geometric endomorphisms of the complex $(E^{\pm}, D^+)$.

Recall the $\widehat{A}$ genus defined in Section 1.

Definition 5.1. *The $\widehat{A}$ genus of F is the Haefliger zero form*

$$\widehat{A}(F) = \int_F \widehat{A}_{k/2}(TF) .$$

In particular, if k is odd, $\widehat{A}(F) = 0$.

Note that we have defined $\widehat{A}(F)$ as the *zero th* order part of $\int_F \widehat{A}(TF)$. For an interpretation of the higher order terms of $\int_F \widehat{A}(TF)$, see [He].

The Lefschetz Theorem for Foliations applied to the case $f = id_M$, $T = id$ says that $\widehat{\mathcal{A}}(F)$ is equal to the index of the leafwise Spin complex, which is just $L(I)$. The Connes Index Theorem [C] says that it is also equal to the index of the holonomy covering leafwise Spin complex.

We now prove the theorem of the introduction, namely

THEOREM 5.2 ([HL 2]). *Let F be an oriented foliation of a compact oriented manifold M and assume that F admits a Spin structure. If a compact connected Lie group acts non-trivially on M as a group of isometries taking each leaf of F to itself and preserving the Spin structure on F, then the $\widehat{A}$ genus of F is zero.*

As a corollary, we have the well known result of Atiyah and Hirzebruch.

THEOREM 5.3 ([AH]). *Let M be a compact connected oriented manifold which admits a Spin structure. If a compact connected Lie group acts non-trivially on M, then $\widehat{\mathcal{A}}(M) = \int_M \widehat{\mathcal{A}}(TM)$ is zero.*

Of course, this theorem and its proof were the inspiration for Theorem 5.2.

Now let G be a compact connected Lie group acting on M by isometries taking each leaf of F to itself and preserving the Spin structure on F. We quote two results from [HL 2] and refer the reader to that paper for the proofs. Note that in [HL 1] and [HL 2], we assume that F admits a transverse invariant measure. A careful reading of those papers shows that in fact we may disregard the invariant transverse measure and consider the traces used as taking values in the Haefliger zero forms of F and all the results remain valid. See the remarks on this in [HL 3].

LEMMA 5.4. *The fixed point set of the action of G is a closed submanifold of M which is transverse to F.*

THEOREM 5.5. *The Lefschetz number $L(g)$ is a continuous function on G.*

Proof of Theorem 5.2. We may assume $G = S^1 \subset \mathbf{C}$. Let N be the fixed point set of G, N_α a connected component of N, L a leaf of F and $y \in N_\alpha \cap L$. The normal bundle to $N_\alpha \cap L$ in L at y can be written as $\oplus V_y^j$, where V_y^j is a complex vector space and $z \in G$ acts on V_y^j by multiplication by z^{m_j} for some positive integer m_j. It follows that the V^j are complex G vector bundles on $N_\alpha \cap L$.

Now let $z \in \mathbf{C}$, $z \neq 1$ and consider the function $R(x, z) = 1/(1 - ze^{-x})$. It can be written as a formal power series in x whose coefficients are rational functions in z having a pole only at $z = 1$, and no pole at $z = \infty$. To see this, write

$$\begin{aligned}\frac{1}{1 - ze^{-x}} = \sum_{k=0}^{\infty}(ze^{-x})^k = \sum_{k=0}^{\infty} z^k e^{-kx} &= (1 + z + z^2 + z^3 + \cdots) \\ &- (z + 2z^2 + 3z^3 + \cdots)x \\ &+ (z + 2^2z^2 + 3^2z^3 + \cdots)x^2/2! \\ &- \cdots .\end{aligned}$$

Set $f_0(z) = 1 + z + z^2 + \cdots = 1/(1 - z)$, and for $n \geq 1$, set $f_n(z) = \sum_{k=1}^{\infty} k^n z^k$. Then $(-1)^n f_n(z)/n!$ is the coefficient of x^n in $R(x, z)$ and it is obvious that $f_{n+1}(z) = zf_n'(z)$. An induction argument then shows that $f_n(z)$ is a rational function of z with a pole only at $z = 1$ and no pole at $z = \infty$. By induction we also have that $z^{1/2} f_n(z)$ has a pole only at $z = 1$ and, as it is $\mathcal{O}(z^{-1/2})$ at $z = \infty$, it has no pole at $z = \infty$.

Now for fixed $z \neq 1$, set $Q(x, z) = z^{1/2} e^{-x/2} R(x, z)$, which is a formal power series in x. Denote the corresponding multiplicative sequence by $B(\ , z) = \big(B_0(\ , z), B_1(\ , z), \ldots\big)$.

Let $z \in G = S^1$ be a topological generator (i.e. z generates a dense subgroup). Then the fixed point set of z is N and z acts on V^j by multiplication by z^{m_j}. Let d_j be the complex dimension of V^j and set

$$B(V^j, z) = B_{d_j}(V^j, z^{m_j}) .$$

$B(V^j, z)$ is a cohomology class on $N_\alpha \cap L$ whose coefficients are rational functions of z having poles only at roots of unity and no pole at $z = \infty$. Set

$$B(N_\alpha \cap L, z) = \prod_j B(V^j, z) .$$

As $B(V^j, z)$ contains the factor $(z^{m_j d_j})^{1/2}$, $B(N_\alpha \cap L)$ contains the factor $(z^d)^{1/2}$, $d = \sum m_j d_j$, and so is defined only up to sign. The choice of sign is determined as in [AH], page 21.

The Riemannian connection on TM over $N_\alpha \cap L$ preserves the bundles V^j and is a complex connection on each V^j. Using this connection and the Riemannian connection on $T(N_\alpha \cap L)$, we may construct the differential form

$w^L_\alpha(z)$ on $N_\alpha \cap L$ which represents the cohomology class $\widehat{\mathcal{A}}(N_\alpha \cap L)B(N_\alpha \cap L, z)$. Then $w^L_\alpha(z)$ is the form a^L_α given in the foliation Lefschetz theorem for z acting on the leafwise Spin complex, and it defines a smooth form $w_\alpha(z)$ on N_α. Thus for $z \in S^1$, z not a root of unity, we have

$$L(z) = \int_N w(z) = \sum_\alpha \int_{N_\alpha} w_\alpha(z) .$$

Now notice that the right side of this equation defines a function $A(F,z)$ on the complex plane with values in the Haefliger forms of F. Also note that $A(F,z)$ has poles only at roots of unity and no pole at $z = \infty$, since $w_\alpha(z)$ has poles only at roots of unity and no pole at $z = \infty$. Because of the factor of $(z^d)^{1/2}$, $A(F,0) = 0$. For $z \in S^1$, z not a root of unity, $A(F,z) = L(z)$. But $L(z)$ is defined for all $z \in S^1$ and by Theorem 5.5 it is continuous on S^1. Thus $A(F,z)$ has no poles at all. Since it is analytic and bounded, it is constant and hence is identically zero. Therefore $L(z) = 0$ for all $z \in S^1$, but $L(1) = \widehat{\mathcal{A}}(F)$ so we are done.

The compactness of G is essential, as in [HL 2], we give an example of an infinite discrete group acting by leaf preserving isometries on a compact oriented foliated manifold M, F and G preserves a Spin structure on F. The foliation F admits an invariant transverse measure which defines a map from the Haefliger zero forms of F to $\mathbf{C}$. The image of $\widehat{\mathcal{A}}(F)$ under this map is non-zero, so $\widehat{\mathcal{A}}(F) \neq 0$.

6. The Rigidity Theorem of Witten

In 1986, Witten [W] predicted rigidity theorems for the indices of certain elliptic operators on manifolds with S^1 actions. The genesis for Witten's conjecture was his study of the Dirac operator on the free loop space $\mathcal{L}M$ (an infinite dimensional manifold) of a Spin manifold M. $\mathcal{L}M$ admits a natural S^1 action whose fixed point set is diffeomorphic to M. The sequences of bundles $R(q)$ and $R'(q)$ described below were derived from the normal bundle of M in $\mathcal{L}M$ and from the formal analogue on $\mathcal{L}M$ of the fixed point formula for the Dirac operator in the finite dimensional case.

Let $D : C^\infty(E_1) \to C^\infty(E_2)$ be an elliptic operator on a compact manifold M and suppose M admits an S^1 action preserving D. Then as noted above, Index(D) is a virtual S^1 module and has a decomposition into a finite sum of irreducible complex one dimensional representations