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BARBIER'S THEOREM FOR THE SPHERE AND THE HYPERBOLIC PLANE

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1. INTRODUCTION

Curves of constant width are a class of plane curves with surprising properties, not the least being the very existence of such curves which are not circles (see ch. 25 of [RT] for an elementary introduction). Also remarkable is the fact that all curves of the same constant width have the same perimeter: this is the content of Barbier's theorem (whose original proof, using probabilistic methods, appears in [B] and is reproduced in [C], pp. 161-163). Here we investigate how Barbier's theorem generalizes to the complete, simply connected surfaces of constant curvature K, which we denote by S_K (thus S_K is the sphere of radius $\frac{1}{\sqrt{K}}$ if $K > 0$; the euclidean plane if $K = 0$; and the hyperbolic plane with curvature K if $K < 0$). The formulas in our main Theorem B are originally due to Blaschke (for $K > 0$) and Santaló (for $K < 0$), but it seems worthwhile to bring them together using a unified differential geometric approach.

Curves of constant width are defined as follows. Let γ be a convex closed Jordan curve on the euclidean plane, and consider the strips Ω bounded by two parallel lines r_0, r_1 such that $\gamma \subseteq \Omega$ and both r_0, r_1 touch γ at some point. We call such sets Ω *enveloping strips* of γ. There is exactly one enveloping strip of γ which is parallel to any given direction. If all enveloping strips of γ have the same width $\mathcal{W}$ then we say γ has *constant width* $\mathcal{W}$.

The following lemma gives an alternative characterization of these curves. For a curve γ and a point p on it, we let $\mathcal{D}(p) = \max\{|p-q| : q \in \gamma\}$.

LEMMA A. *A simple closed curve γ is of constant width $\mathcal{W}$ if and only if $\mathcal{D}(p) = \mathcal{W}$ for all $p \in \gamma$.*

This lemma allows us to define curves of constant width in metric spaces other than the euclidean plane, and in particular we consider them in regular (differentiable) surfaces endowed with a riemannian metric and associated intrinsic distance. We prove the following result extending Barbier's theorem to S_K.

THEOREM B ([Bl], [S]). *If a curve of constant width $\mathcal{W}$ in S_K has perimeter $\mathcal{L}$ and bounds a region of area $\mathcal{A}$ then*

$$\mathcal{L} = H(K, \mathcal{W}) \{2\pi - K\mathcal{A}\},$$

where $H(K, \mathcal{W})$ is given by

$$\frac{1}{\sqrt{K}} \tan\left(\frac{\sqrt{K}\,\mathcal{W}}{2}\right) \text{ if } K > 0; \frac{\mathcal{W}}{2} \text{ if } K = 0; \frac{1}{\sqrt{-K}} \tanh\left(\frac{\sqrt{-K}\,\mathcal{W}}{2}\right) \text{ if } K < 0.$$

There exists a generalization for S_K of the well-known isoperimetric inequality for the euclidean plane. It states that if a curve γ in S_K of perimeter $\mathcal{L}$ bounds a region of area $\mathcal{A}$ then

$$\mathcal{L}^2 \geq 4\pi\mathcal{A} - K\mathcal{A}^2, \tag{1}$$

and that the equality holds if and and only if γ is a (geodesic) circle (see [O] for a proof). From (1) and the above theorem we obtain the following corollary:

COROLLARY C. *If $K > 0$ (respectively $K < 0$) then, of all the curves γ in S_K with the same constant width $\mathcal{W}$, the circle has the least (resp. the greatest) perimeter. More precisely, we have*

$$\mathcal{L} \geq \frac{2\pi}{\sqrt{K}} \sin\left(\frac{\sqrt{K}\,\mathcal{W}}{2}\right) \qquad \left[\textit{resp. } \mathcal{L} \leq \frac{2\pi}{\sqrt{-K}} \sinh\left(\frac{\sqrt{-K}\,\mathcal{W}}{2}\right)\right],$$

and equality holds if and only if γ is a circle.

From Theorem B we see that, for fixed $\mathcal{W}$ and $K < 0$, curves of longer perimeter enclose larger areas, whereas for $K > 0$ larger areas correspond to shorter perimeters. Thus Corollary C can be expressed by saying that, in all cases, the curve of a given width enclosing the largest area is the circle (for $K = 0$ this follows immediately from combining Barbier's theorem with the isoperimetric inequality).

If γ is a curve of constant width $\mathcal{W}$ in S_K, we say that $p, \tilde{p} \in \gamma$ are *antipodal points* if the (intrinsic) distance between them is $\mathcal{W}$, which is to say that they realize the diameter of γ. We prove a result that was already known in the case of the euclidean plane (see [HS]):

THEOREM D. *If γ is a curve of constant width $\mathcal{W}$ in S_K such that every pair of antipodal points divides γ into two arcs of equal length (and, in the case of the sphere, if $\mathcal{W} < \frac{\pi}{\sqrt{K}}$) then γ is a circle.*

We must emphasize that, except for Lemma A, proofs are only given for *regular curves*, which for us means that they have no corners and the natural parametrization by arc-length is C^∞ (or just C^k for big enough k). By the expedient of using parallel curves, as explained in the next section, we can extend our results to curves consisting of regular pieces and a *finite* number of corners *(piecewise regular curves)*, but further extension does not seem possible using our methods.

The remainder of this article is organized as follows. In the next section we discuss curves of constant width in the familiar setting of the euclidean plane, and prove Lemma A and Barbier's theorem. Our proof of Barbier's theorem is similar to that in section 1.13 of [St], but we choose to present it here since the proof we give for S_K $(K \neq 0)$ is an elaboration of our proof for S_0.

In §3 we consider general oriented surfaces and construct systems of *geodesic parallel coordinates* suitable for dealing with our curves, proving a number of technical results about these coordinates, and also proving Theorem D. In the last section all pieces are put together to give the proofs of Theorem B and Corollary C.

2. CURVES OF CONSTANT WIDTH IN THE EUCLIDEAN PLANE

We now review some background on convex curves; the basic reference here is [E]. Given a closed curve $\gamma \subseteq \mathbf{R}^2$, a straight line r is called a *supporting line* of γ if r touches γ at some point and γ is entirely contained in one of the closed half-planes bounded by r. One possible characterization of convex curves is the following: γ is convex if and only if through every point of γ there passes a supporting line of γ. If some boundary line of an enveloping strip Ω of γ touches γ at the point p then we say Ω is *supported* on p (of course Ω is supported on at least two points of γ).

Proof of Lemma A. The diameter of γ is by definition

$$\mathcal{D} = \max\{|p-q| : p, q \in \gamma\} = \max\{\mathcal{D}(p) : p \in \gamma\}.$$

Let p_0, p_1 be any two points in γ realizing its diameter ($|p_0 - p_1| = \mathcal{D}$), and let r_0, r_1 be the lines through p_0, p_1 which are orthogonal to the segment $\overline{p_0p_1}$. Then the set bounded by r_0, r_1 is an enveloping strip of Ω and has width $\mathcal{D}$.

Now assume γ has constant width $\mathcal{W}$. We have just shown that $\mathcal{W} = \mathcal{D}$. Given $p \in \gamma$, choose a supporting line r_0 through p, and let r_1 be the other supporting line parallel to r_0, touching γ at the point q. Then $|p-q| \geq \mathcal{W}$ (for the distance between r_0 and r_1 is $\mathcal{W}$) and hence $\mathcal{D}(p) \geq \mathcal{W}$. But we also have $\mathcal{D} \geq \mathcal{D}(p)$, and from these inequalities we obtain $\mathcal{D}(p) = \mathcal{W}$.

Now we prove the "if part". Given $p \in \gamma$, let $\tilde{p} \in \gamma$ be such that $|p - \tilde{p}| = \mathcal{W}$. Then p, $\tilde{p}$ realize the diameter of γ, and therefore the enveloping strip Ω orthogonal to $\overline{p\tilde{p}}$ has width $\mathcal{W}$. If γ has a well-defined tangent at p (i.e., if γ is *smooth* at p) then Ω is the only enveloping strip supported on p. Otherwise p is a corner of γ and the supporting lines at p vary between two extreme positions, the "left" and "right" tangents r^l and r^r, which are the limiting positions of the tangents to γ at p_n^l and p_n^r as $(p_n^l)_{n\geq 1}$ and $(p_n^r)_{n\geq 1}$ approach p from the left and from the right, respectively (convexity ensures that the points at which γ is smooth are dense in γ).

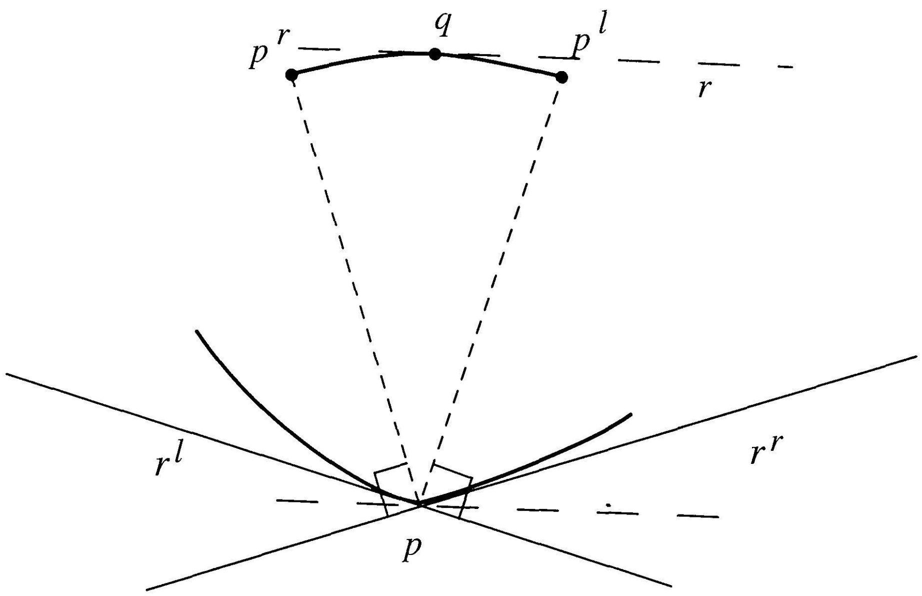

FIGURE 1

To each p_n^l there corresponds a point $\tilde{p}_n^l \in \gamma$ such that $|p_n^l - \tilde{p}_n^l| = \mathcal{W}$ and which is situated along the normal to γ at p_n^l. Therefore $p^l = \lim \tilde{p}_n^l$ is a point of γ such that $|p - p^l| = \mathcal{W}$ and which is on the line through p orthogonal to r^l; and similarly for a point p^r on the line orthogonal to r^r. It follows that the lines through p^l and p^r parallel to r^l and r^r, respectively, are supporting lines of γ. Now take an interior point q in the arc $\widehat{p^l p^r}$ of γ opposite p, and consider any supporting line r of γ at q: this line r is parallel to some supporting line through p. Hence every enveloping strip supported on q must also be supported on p, and it follows that p is the point of γ at the maximum distance $\mathcal{W}$ from q. Therefore $\widehat{p^l p^r}$ is an arc of circle with centre p and radius $\mathcal{W}$. Hence all enveloping strips supported on p have width $\mathcal{W}$.

We have thus shown that all enveloping strips of γ have width $\mathcal{W}$. □

The important point to retain from the proof of Lemma A is that, if we choose a supporting line r at a point p of a curve γ of constant width $\mathcal{W}$, then the perpendicular to r through p intersects γ at a point $\tilde{p}$ such that $|p - \tilde{p}| = \mathcal{W}$ — i.e., at an antipodal point of p.

For completeness, we observe that if γ is a simple closed curve such that $\mathcal{D}(p) = \mathcal{W}$ for all p on γ then γ is convex. For, given $p \in \gamma$, choose $\tilde{p} \in \gamma$ such that $|p - \tilde{p}| = \mathcal{W}$: γ is entirely contained in the closed disk $\mathbf{D}$ with center $\tilde{p}$ and radius $\mathcal{W}$, and the circumference $\mathcal{C}$ of $\mathbf{D}$ touches γ at the point p. Hence the tangent to $\mathcal{C}$ at p is a supporting line of γ, and we conclude that through every point of γ there passes a supporting line. This means γ is convex.

If γ is twice differentiable at p then the preceding argument shows that the absolute value $|k(p)|$ of the curvature of γ at p is *not less* than $\frac{1}{\mathcal{W}}$ (which is the curvature of $\mathcal{C}$).

For the rest of this section we consider a regular curve $\gamma(s)$ of constant width $\mathcal{W}$ and perimeter $\mathcal{L}$, parametrized by the arc length s. We define $\gamma(s)$ for all $s \in \mathbf{R}$ by letting $\gamma(s + \mathcal{L}) = \gamma(s)$, and assume that γ is traversed in the counterclockwise direction.

Let $\varphi(s)$, $s \in \mathbf{R}$, be a differentiable determination of the oriented angle between the positive x-axis and the tangent vector $\gamma'(s)$. This simply means that $\gamma'(s) = \big(\cos\varphi(s), \sin\varphi(s)\big)$. The tangent vector $\gamma'(s)$ completes one counterclockwise revolution as the point $\gamma(s)$ travels once around γ, which implies that $\varphi(s + \mathcal{L}) = \varphi(s) + 2\pi$. Since γ is convex, φ is non-decreasing, so that $\varphi'(s) \geq 0$ for all $s \in \mathbf{R}$. The *signed curvature* of γ at the point $\gamma(s)$

is given by $k(s) = \varphi'(s)$, and so we have

$$\int_0^{\mathcal{L}} k(s)\, ds = \varphi(\mathcal{L}) - \varphi(0) = 2\pi. \tag{2}$$

Now define the normal vector $n(s) = \big(-\sin\varphi(s), \cos\varphi(s)\big)$, so that for each s the pair $\big(\gamma'(s), n(s)\big)$ is a positively oriented orthonormal basis of $\mathbf{R}^3$. The following formulas (special cases of Frenet's formulas) are readily verified:

$$\gamma''(s) = k(s)\, n(s), \qquad n'(s) = -k(s)\, \gamma'(s). \tag{3}$$

Let us indicate by $\Pi\big(\gamma(s)\big)$ the antipodal point of $\gamma(s)$. This map Π is an *involution* of the curve γ, in the sense that $\Pi \circ \Pi$ is the identity. By the above discussion we have

$$\Pi \circ \gamma(s) = \gamma(s) + \mathcal{W}\, n(s), \tag{4}$$

which shows that $s \mapsto \Pi \circ \gamma(s)$ is differentiable. Thus Π is an orientation-preserving diffeomorphism of the curve onto itself (if the orientation were reversed then Π would have a fixed point, which it does not). As in the case of circle diffeomorphisms, there exists a continuous function $f\colon \mathbf{R} \to \mathbf{R}$ that "lifts" Π — i.e., that satisfies the equality $\Pi \circ \gamma = \gamma \circ f$.

LEMMA E. *There exists a diffeomorphism $f\colon \mathbf{R} \to \mathbf{R}$ such that*

$$\Pi \circ \gamma(s) = \gamma \circ f(s),$$
$$f(s + \mathcal{L}) = f(s) + \mathcal{L}$$

for all $s \in \mathbf{R}$. All other liftings of Π are of the form $f + n\mathcal{L}$ for some $n \in \mathbf{Z}$.

In the case of circle diffeomorphisms, which is in essence no different from the one above, this is a standard result, and so we omit the proof of Lemma E. The uniqueness assertion has, however, some interesting consequences. For instance, from the equality $(\Pi \circ \Pi) \circ \gamma = \Pi \circ \gamma \circ f = \gamma \circ (f \circ f)$ it follows that $f \circ f(s) = s + n\mathcal{L}$, since both $f \circ f$ and the identity function are liftings of the diffeomorphism $\Pi \circ \Pi = id$.

We now prove Barbier's theorem. By Lemma E, we can rewrite (4) as $\gamma \circ f(s) = \gamma(s) + \mathcal{W}\, n(s)$. Taking the derivative of both sides and using Frenet's formulas (3) we get

$$f'(s)\, \gamma'\big(f(s)\big) = \{1 - \mathcal{W}\, k(s)\}\, \gamma'(s).$$

Since $\gamma'\bigl(f(s)\bigr)$ and $\gamma'(s)$ are unit vectors pointing in opposite directions, this implies $f'(s) = \mathcal{W}\,k(s) - 1$. Integrating and using Lemma E and (2), we finally have

$$\begin{aligned}\mathcal{L} = f(\mathcal{L}) - f(0) &= \int_0^{\mathcal{L}} f'(s)\,ds \\ &= -\mathcal{L} + \mathcal{W}\int_0^{\mathcal{L}} k(s)\,ds = -\mathcal{L} + 2\pi\mathcal{W},\end{aligned}$$

which concludes the proof.

When we allow piecewise regular curves, the antipodal map is no longer well-defined, let alone being a diffeomorphism — and the above approach fails to work. Thus so far our proof of Barbier's theorem does not even include the simplest non-circular curve of constant width : the Reuleaux triangle.

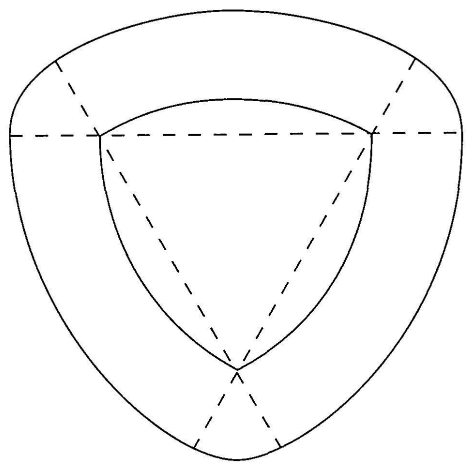

FIGURE 2

The Reuleaux triangle with one of its parallel curves

One way to overcome this deficiency is to consider *parallel curves.* If γ is a convex curve then its parallel curve γ_d surrounds γ at a fixed distance $d > 0$ from it. Assuming γ has constant width $\mathcal{W}$, γ_d has constant width $\mathcal{W} + 2d$. When γ is piecewise differentiable then each of its corners is replaced by an arc of circle with radius d in γ_d ; and γ_d no longer has corners, possessing a continuously turning tangent and a continuous, piecewise differentiable angular determination $\varphi_d(s)$. With minor adaptations, our proof of Barbier's theorem shows that the perimeter of γ_d is given by

$$\mathcal{L}_d = \pi(\mathcal{W} + 2d)$$

— and letting $d \to 0$ we obtain $\mathcal{L} = \pi\mathcal{W}$ as we want.

3. Geodesic parallel coordinates

In this section we consider an oriented, connected, C^∞ surface S. For the sake of simplicity, we assume S is embedded in $\mathbf{R}^3$ whenever convenient (of course the hyperbolic plane cannot be embedded in $\mathbf{R}^3$, but our arguments have a local character, involving only the computation of derivatives; and there are surfaces in $\mathbf{R}^3$ which are locally isometric to the hyperbolic plane).

We consider a regular closed Jordan curve $\gamma(s)$ in S of constant width $\mathcal{W}$. If $\mathcal{L}$ is the perimeter of the curve, we extend $\gamma(s)$ periodically by setting $\gamma(s+\mathcal{L}) = \gamma(s)$.

We would like to say that the antipodal point $\tilde{p}$ of p is situated along the geodesic that cuts γ orthogonally at p. But some care is necessary, and we make an extra assumption on γ :

Standing Assumption (**SA**). *There exists $\varepsilon > 0$ such that, for every $p \in \gamma$, the restriction of the exponential map* $\exp_p$ *to*

$$\{v \in T_pS : \|v\| < \mathcal{W} + \varepsilon\}$$

is a diffeomorphism onto its image.

Condition **SA** ensures that there is exactly one minimizing geodesic between any two points of γ, and that γ is indeed the boundary of some Jordan region in S. On the hyperbolic plane, **SA** represents no restriction whatever, whereas on the sphere of radius $\frac{1}{\sqrt{K}}$ it is equivalent to the requirement that $\mathcal{W} < \frac{\pi}{\sqrt{K}}$ — and this is no strong restriction either, for in any case we would have $\mathcal{W} \leq \frac{\pi}{\sqrt{K}}$, since the maximum (intrinsic) distance between distinct points on the sphere is $\frac{\pi}{\sqrt{K}}$.

Claim 1. *If a curve γ of constant width $\mathcal{W}$ satisfies* **SA** *then the minimizing geodesic between any pair of antipodal points p, $\tilde{p}$ intersects γ orthogonally at both p and $\tilde{p}$.*

Proof. Take a system of geodesic polar coordinates $\Phi(\rho,\theta)$ centered at $\tilde{p}$. If $\gamma(s_0) = p$ then there exists $\delta > 0$ such that, for $s \in\,]s_0 - \delta, s_0 + \delta[$, we can write $\gamma(s) = \Phi\big(\rho(s), \theta(s)\big)$ for some differentiable functions $\rho(s)$, $\theta(s)$. Our assumption implies that $\rho(s_0) = \mathcal{W}$ and $\rho(s) \leq \mathcal{W}$ for all s, and therefore $\rho'(s_0) = 0$. Hence $\gamma'(s_0) = \theta'(s_0)\,\Phi_\theta$, which implies that γ and the radial (minimizing) geodesic from $\tilde{p}$ to p cut each other orthogonally at p. Reversing the roles of p and $\tilde{p}$ we show that the intersection at $\tilde{p}$ is also orthogonal. $\square$

We have just observed that if p, $\tilde{p}$ are antipodal points of γ then γ is inside the geodesic circle $\mathcal{C}(\tilde{p}, \mathcal{W})$ of centre $\tilde{p}$ and radius $\mathcal{W}$, and touches it at the point p. As in the euclidean case, the geodesic curvature of $\mathcal{C}(\tilde{p}, \mathcal{W})$ at p is a lower bound for the geodesic curvature of γ at p, as we now proceed to show. We assume that both curves are traversed counterclockwise (i.e., the Jordan region bounded by γ is always to our left as we move around γ), and recall that the coeficients E, F, G of the first fundamental form of $\Phi(\rho, \theta)$ are such that $E \equiv 1$ and $F \equiv 0$ (see [dC], p. 287).

CLAIM 2. *Let the coordinates of p be $\rho = \mathcal{W}$, $\theta = \theta_0$, and denote by $k_g(p)$ and $\tilde{k}_g(p)$ the geodesic curvatures at p of γ and $\mathcal{C}(\tilde{p}, \mathcal{W})$, respectively. Then we have*

$$k_g(p) \geq \tilde{k}_g(p) = \left.\frac{G_\rho}{2G}\right|_{(\mathcal{W}, \theta_0)} .$$

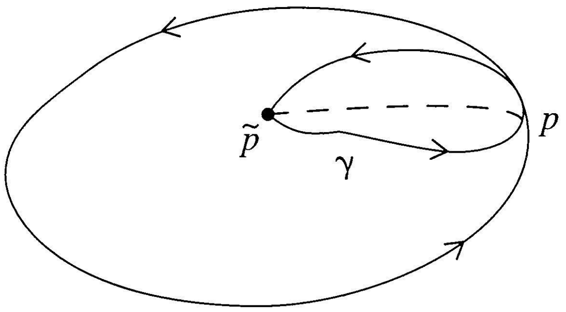

FIGURE 3

Proof. We can reparametrize γ in a neighbourhood of p by setting $\gamma(t) = \Phi\big(\rho(t), \theta_0 + t\big)$ for $t \in]-\delta, \delta[$. Thus $\rho(0) = \mathcal{W}$ and, as in the proof of Claim 1, $\rho(t)$ attains a maximum at $t = 0$, so that $\rho'(0) = 0$ and $\rho''(0) \leq 0$. The geodesic curvature of γ at $\gamma(t)$ is given by

$$k_g(t) = \frac{1}{\|\gamma'(t)\|^2} \big\langle \gamma''(t), n(t) \big\rangle ,$$

where $n(t)$ is the *unit* vector such that $\big(\gamma'(t), n(t)\big)$ is a positively oriented *orthogonal* basis of $T_{\gamma(t)}S$. We have $\gamma'(0) = \Phi_\theta$, and therefore $n(0) = -\Phi_\rho$, $\|\gamma'(0)\|^2 = G$, and

$$k_g(p) = k_g(0) = \frac{1}{G} \big\langle \gamma''(0), -\Phi_\rho \big\rangle .$$

Since $\gamma''(0) = \rho''(0)\, \Phi_\rho + \Phi_{\theta\theta}$, it follows that

$$k_g(p) = -\frac{1}{G}\rho''(0) - \frac{1}{G}\langle \Phi_{\theta\theta}, \Phi_\rho \rangle \geq -\frac{1}{G}\langle \Phi_{\theta\theta}, \Phi_\rho \rangle \cdot \tag{5}$$

Our calculations also show that the right-hand side of (5) is just $\tilde{k}_g(p)$. By taking the derivative with respect to θ of the equality $\langle \Phi_\theta, \Phi_\rho \rangle \equiv 0$, we obtain $\langle \Phi_{\theta\theta}, \Phi_\rho \rangle = -\frac{1}{2}G_\rho$ — and this, together with (5), proves our claim. □

At this point we recall ([dC], p. 289) that in S_K the coefficient G is given by

$$\frac{1}{K}\sin^2(\sqrt{K}\rho) \text{ if } K > 0; \qquad \rho^2 \text{ if } K = 0; \qquad -\frac{1}{K}\sinh^2(\sqrt{-K}\rho) \text{ if } K < 0$$

— and thus in S_K Claim 2 reduces to:

The geodesic curvature $k_g(s)$ of a curve of constant width $\mathcal{W}$ in S_K is such that

$$k_g(s) \geq F(K, \mathcal{W}), \tag{6}$$

where $F(K, \mathcal{W})$ is given by

$$\frac{\sqrt{K}\cos(\sqrt{K}\,\mathcal{W})}{\sin(\sqrt{K}\,\mathcal{W})} \text{ if } K > 0; \quad \frac{1}{\mathcal{W}} \text{ if } K = 0; \quad \frac{\sqrt{-K}\cosh(\sqrt{-K}\,\mathcal{W})}{\sinh(\sqrt{-K}\,\mathcal{W})} \text{ if } K < 0 .$$

Notice that we do not necessarily have $k_g(s) \geq 0$: for $K > 0$ and $\mathcal{W} > \frac{\pi}{2\sqrt{K}}$, the lower bound in (6) is negative. Related to this is the fact that not all curves of constant width in the sphere are convex (see the remark at the end of this section).

Now we let $n(s)$ be the *unit* vector field along $\gamma(s)$ which is orthogonal to $\gamma'(s)$ and points to the interior of the region bounded by γ, so that $(\gamma'(s), n(s))$ is positively oriented. If we travel a distance $\mathcal{W}$ along the geodesic $t \mapsto \exp_{\gamma(s)}(t\,n(s))$ we reach the antipodal point $\Pi(\gamma(s))$ of $\gamma(s)$. In other words,

$$\Pi \circ \gamma(s) = \exp_{\gamma(s)}(\mathcal{W}\,n(s)) . \tag{7}$$

It is only natural to consider the map $\Psi(t, s) = \exp_{\gamma(s)}(-t\,n(s))$, where the minus sign ensures that (Ψ_t, Ψ_s) is positively oriented for small t. This is not really a parametrization, since it is not injective and may have singularities. We define the coefficients $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ by

$$\mathcal{E} = \langle \Psi_t, \Psi_t \rangle, \quad \mathcal{F} = \langle \Psi_t, \Psi_s \rangle, \quad \mathcal{G} = \langle \Psi_s, \Psi_s \rangle \cdot$$

CLAIM 3. *The following equalities hold:* $\mathcal{E} \equiv 1$; $\mathcal{F} \equiv 0$; $\mathcal{G}(0, s) = 1$ *for all* $s \in \mathbf{R}$.

Proof. For fixed s, the curve $t \mapsto \Psi(t, s)$ is a geodesic parametrized with constant speed $\|n(s)\| = 1$, and therefore $\mathcal{E} \equiv 1$. The third equality is obvious. To prove the second one, we observe that $\mathcal{F}(0, s) = \langle -n(s), \gamma'(s) \rangle = 0$ and that

$$\frac{\partial \mathcal{F}}{\partial t} = \langle \Psi_{tt}, \Psi_s \rangle + \langle \Psi_t, \Psi_{st} \rangle = \left\langle \frac{D\Psi_t}{\partial t}, \Psi_s \right\rangle + \frac{1}{2} \frac{\partial \mathcal{E}}{\partial s} = 0,$$

where $\dfrac{D\Psi_t}{\partial t}$ denotes the covariant derivative of the "velocity" vector field $t \mapsto \Psi_t(t, s)$ along the geodesic $t \mapsto \Psi(t, s)$ (which is identically zero by the definition of a geodesic). □

In the neighbourhood of any point (t, s) where $\mathcal{G}$ is non-zero, the map Ψ is a true parametrization, and by Claim 3 its coefficients $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ are analogous to the coefficients E, F, G of the geodesic polar coordinates. Thus the proof of Claim 2 shows that, provided $\Psi(t, s)$ agrees with the orientation of S, the geodesic curvature of the curves $t = \text{constant}$ is given by

$$\frac{\mathcal{G}_t}{2\mathcal{G}} = \frac{(\sqrt{\mathcal{G}})_t}{\sqrt{\mathcal{G}}};$$

in particular, setting $t = 0$ and using Claim 3, we obtain

$$(\sqrt{\mathcal{G}})_t(0, s) = k_g(s). \tag{8}$$

There exists a very useful formula for the Gaussian curvature K in terms of the coefficients of an orthogonal parametrization ([dC], p. 237), which in this case simplifies to

$$(\sqrt{\mathcal{G}})_{tt} + K\sqrt{\mathcal{G}} = 0.$$

This formula holds whenever $\mathcal{G}(t, s) \neq 0$. Turning our attention to S_K, $t \mapsto \sqrt{\mathcal{G}}(t, s)$ is then the solution of the differential equation

$$x''(t) + K\,x(t) = 0 \tag{9}$$

which, by Claim 3 and (8), satisfies the initial conditions $x(0) = 1$ and $x'(0) = k_g(s)$. Thus we find that $\sqrt{\mathcal{G}}(t, s)$ is given by:

$$\cos(\sqrt{K}\,t) + \frac{k_g(s)}{\sqrt{K}} \sin(\sqrt{K}\,t) \qquad \text{if } K > 0, \tag{10}$$

$$1 + t\,k_g(s) \qquad \text{if } K = 0, \tag{11}$$

$$\cosh(\sqrt{-K}\,t) + \frac{k_g(s)}{\sqrt{-K}} \sinh(\sqrt{-K}\,t) \qquad \text{if } K < 0. \tag{12}$$

We are running into trouble here: formulas (10)-(12) may assume negative values for $t \neq 0$, and $\sqrt{\mathcal{G}}$ is necessarily non-negative. However, we must keep in mind that in any case $\mathcal{G}$ is a differentiable (C^∞) function, as its definition ensures. The only way to reconcile this with the fact that $t \mapsto \sqrt{\mathcal{G}}(t,s)$ is a solution of (9) whenever $\mathcal{G}(t,s) \neq 0$ is that $\mathcal{G}(t,s)$ be equal to the *square* of formulas (10)-(12) for all (t,s).

Let f be a lifting of the antipodal map Π as in Lemma E. We can rewrite (7) as

$$\gamma \circ f(s) = \Psi(-\mathcal{W}, s).$$

Taking the derivative of both sides we obtain $f'(s)\,\gamma'\big(f(s)\big) = \Psi_s|_{(-\mathcal{W},s)}$, and from here we get

$$[f'(s)]^2 = \mathcal{G}(-\mathcal{W}, s). \tag{13}$$

The reader should now check that inequality (6) yields that, for $t = -\mathcal{W}$, each of the formulas (10)-(12) is non-positive for all $s \in \mathbf{R}$. Since $f'(s) > 0$, (13) and the above discussion imply that $f'(s)$ is equal to

$$\frac{k_g(s)}{\sqrt{K}} \sin(\sqrt{K}\,\mathcal{W}) - \cos(\sqrt{K}\,\mathcal{W}) \qquad \text{if } K > 0, \tag{14}$$

$$\mathcal{W}\,k_g(s) - 1 \qquad \text{if } K = 0, \tag{15}$$

$$\frac{k_g(s)}{\sqrt{-K}} \sinh(\sqrt{-K}\,\mathcal{W}) - \cosh(\sqrt{-K}\,\mathcal{W}) \qquad \text{if } K < 0. \tag{16}$$

Formula (15) was already known from §2. In the next section we use formulas (14) and (16) to prove Theorem B and Corollary C. As an appetizer we now prove Theorem D.

Proof of Theorem D. This is a simple consequence of the uniqueness part of Lemma E. Under our hypothesis, a possible lifting of Π is $f(s) = s + \frac{1}{2}\mathcal{L}$, and therefore $f'(s) = 1$ for all $s \in \mathbf{R}$. Each of the formulas (14)-(16) then implies that the geodesic curvature k_g of γ is constant. Substituting the value of k_g in (10)-(12) we find that $\mathcal{G}(-\frac{1}{2}\mathcal{W}, s) = 0$ for all $s \in \mathbf{R}$. Hence $s \mapsto \Psi(-\frac{1}{2}\mathcal{W}, s)$ is constant, say equal to p, and therefore γ is the geodesic circle $\mathcal{C}(p, \frac{1}{2}\mathcal{W})$. $\square$

REMARK. We have so far excluded from our discussion curves of constant width $\frac{\pi}{\sqrt{K}}$ on the sphere $x^2+y^2+z^2=\frac{1}{K}$ $(K>0)$. Although our methods do not apply to these curves, they are easily dealt with, being characterized as the Jordan curves γ which remain invariant under the isometry $g\colon S_K \to S_K$ given by $g(x,y,z)=(-x,-y,-z)$. This map g exchanges the two regions bounded by γ in S_K (so these regions have the same area $\frac{2\pi}{K}$), and also exchanges the two arcs into which γ is divided by any pair of antipodal points (so these two arcs have the same length). Hence Theorem D is not valid in this case. If we consider (for small d) a parallel curve γ_d to γ then γ has constant width $\frac{\pi}{\sqrt{K}}-2d$. Since γ has arbitrarily long perimeter and does not need to be convex, the same applies to γ_d (but the longer the perimeter of γ, the smaller d must be in order to ensure that γ_d has no self-intersections).

4. PROOF OF THE MAIN RESULTS

We have now gathered all the necessary tools, and the proofs of Theorem B and Corollary C are a simple matter.

Proof of Theorem B. We assume $K>0$, the case $K<0$ being similar. Using (14) we have

$$\mathcal{L}=f(\mathcal{L})-f(0)=\int_0^{\mathcal{L}} f'(s)\,ds=\frac{\sin(\sqrt{K}\,\mathcal{W})}{\sqrt{K}}\int_0^{\mathcal{L}} k_g(s)\,ds-\mathcal{L}\cos(\sqrt{K}\,\mathcal{W}),$$

and therefore

$$\begin{aligned}\mathcal{L}&=\frac{\sin(\sqrt{K}\,\mathcal{W})}{\sqrt{K}\left\{1+\cos(\sqrt{K}\,\mathcal{W})\right\}}\int_0^{\mathcal{L}} k_g(s)\,ds\\&=\frac{1}{\sqrt{K}}\tan\left(\frac{\sqrt{K}\,\mathcal{W}}{2}\right)\{2\pi-K\mathcal{A}\}\end{aligned}$$

by the Gauss-Bonnet theorem. □

Proof of Corollary C. First we treat the case $K>0$. From Theorem B we see that $\mathcal{A}<\frac{2\pi}{K}$, which means that the region we are interested in has the smallest area of the two regions bounded by γ in S_K. We also assume that $\mathcal{L} \leq \frac{2\pi}{\sqrt{K}}$, otherwise $\mathcal{L}$ is too large for γ to be a circle and the desired inequality holds trivially. Under these conditions inequality (1) is equivalent to

$$\mathcal{A} \leq \frac{1}{K}\left\{2\pi - \sqrt{4\pi^2 - K\mathcal{L}^2}\right\}. \tag{17}$$

Combining Theorem B and (17) we obtain

$$\mathcal{L} \geq \frac{1}{\sqrt{K}} \tan\left(\frac{\sqrt{K}\mathcal{W}}{2}\right)\sqrt{4\pi^2 - K\mathcal{L}^2},$$

which is equivalent to

$$\mathcal{L} \geq \frac{2\pi}{\sqrt{K}} \sin\left(\frac{\sqrt{K}\mathcal{W}}{2}\right) \tag{18}$$

— and this is the inequality we want. If equality holds in (18) then it also holds in each of the equivalent inequalities (17) and (1) — and therefore γ is a circle.

The case $K < 0$ has a similar (and easier) treatment. We begin by rewriting (1) in the form

$$\mathcal{A} \leq -\frac{1}{K}\left\{\sqrt{4\pi^2 - K\mathcal{L}^2} - 2\pi\right\},$$

and then proceed as before. $\square$

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