3. SUFFICIENCY FOR A SINGLE HOMOLOGY CLASS

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homology class in F, up to sign. It follows, therefore, from the classification of surfaces, that each X_ℓ has Euler characteristic ≤ -1. Therefore, when $n > 1$ and $n = g$,

$$\chi(F) = \chi(\bar{F}) = 2 - 2n = \sum \chi(X_\ell) \leq (\text{card } S - n + 1)(-1)$$

or equivalently card $S \leq 3n - 3$, as required.

It remains to consider the case when $g > n > 1$. In this case, we first proceed as before, cutting open along the A_i, obtaining a connected surface $\widehat{F}$ of genus $g - n$ and with $2n$ boundary curves, containing the m curves C_j, each of which is homologous to a sum of boundary curves in $\widehat{F}$. Now each of the C_j separates $\widehat{F}$, and we may further cut open along the C_j, obtaining a surface with $m+1$ components and total genus $g - n$. It follows that there are additional pairwise disjoint simple closed curves E_k, $k = 1, \ldots, g - n$, in $\widehat{F}$, reducing $\widehat{F}$ to a planar surface of genus 0 when we cut open along the E_k and cap off the resulting $2(g - n)$ boundary curves with disks. Call this latter surface $\bar{F}$, topologically a 2-sphere with $2n$ holes. Now the C_j separate $\bar{F}$ into $m + 1 =$ card $S - n + 1$ planar components X_ℓ. As before, each X_ℓ has Euler characteristic ≤ -1. Therefore, again,

$$\chi(\bar{F}) = 2 - 2n = \sum \chi(X_\ell) \leq (\text{card } S - n + 1)(-1)$$

or equivalently card $S \leq 3n - 3$, as required.

3. Sufficiency for a single homology class

Here we collect some basic information about the embedding of a single simple closed curve in a surface, and offer an alternative, elementary proof of Theorem 2 for the well-known case of a single homology class.

LEMMA 3.1. *A nonzero homology class $\alpha \in H_1(F)$ is primitive if and only if there exists $\gamma \in H_1(F)$ such that $\gamma \cdot \alpha = 1$.*

Proof. A nonzero element of a finitely generated free abelian group is primitive if and only if it is part of a basis if and only if there is a $\mathbf{Z}$-valued homomorphism that takes the value 1 on it. Recall that taking intersection numbers of 1-cycles defines a skew symmetric bilinear form on $H_1(F)$. The content of Poincaré Duality in this situation is that this bilinear form is nonsingular, that is, the adjoint homomorphism $H_1(F) \to \operatorname{Hom}\big(H_1(F), \mathbf{Z}\big)$ is an isomorphism. The lemma then follows.

LEMMA 3.2. *Any homology class $\alpha \in H_1(F)$ can be represented by an immersed, oriented closed curve on F and also by an embedded, oriented 1-submanifold.*

Proof. The Hurewicz homomorphism $\pi_1(F) \to H_1(F)$ is onto. Compare W. Massey [1980], Chapter III, Section 7, for example. Any map $S^1 \to F$ can be approximated by an immersion, with only isolated double points. One can surger any double points, that is, one can replace any pair of small oriented arcs having a single transverse intersection with a pair of parallel oriented arcs with the same end points and lying within a regular neighborhood of the intersecting arcs. In this way one creates a disjoint union of oriented simple closed curves representing the same homology class.

PROPOSITION 3.3. *A homology class α in $H_1(F)$ can be represented by a simple closed curve on F if and only if α is primitive.*

Proof. We sketch a 2-dimensional version of the argument of Bennequin [1977]. If a simple closed curve represents a nonzero homology class, then it is nonseparating. It follows that there is a simple closed curve that meets it transversely in a single point. This implies indivisibility, by the homology invariance of intersection numbers.

For the converse, we may assume that α is nonzero. We begin by representing α by a disjoint union A of oriented simple closed curves, as in Lemma 3.2. We shall assume that A contains the smallest possible number of components and show that this number can be reduced unless it is 1 or it is equal to the divisibility of α.

Cut open F along A–that is, remove the interior of a small tubular neighborhood of A. The boundary of the cut open surface $\widehat{F}$ consists of two copies A_i^+ and A_i^- of each component A_i of A, each of which we orient as the boundary of the orientable surface $\widehat{F}$. The positive components A_i^+ have the same orientation as A_i, while the negative components A_i^- have the opposite orientation.

If some component R of $\widehat{F}$ contains in its boundary two positive curves A_i^+ and A_j^+ (or two negative curves), then they can be banded together in an orientable way using a band in R. That is, one chooses an embedded arc δ in R meeting A_i^+ and A_j^+ in its two end points only. One then replaces A_i and A_j with the single simple closed curve obtained by removing small arcs in A_i and A_j containing the end points of δ and inserting in their place two embedded arcs parallel to δ. This would reduce the number of components of A. If some component R has boundary just A_i^+ and A_i^- for some A_i, then we can conclude that A is connected and we are done. If some R has more than two boundary components, then it contains two positive curves or two negative curves, and we can proceed as above to reduce the number of components of A.

It remains to consider the case where each component R_k of $\widehat{F}$ has exactly two boundary components of the form A_i^+ and A_j^-, where A_i and A_j are distinct components of A. In this case we conclude that we can arrange the components of A in a sequence $A_1, A_2, \ldots, A_n$, so that A_1 is homologous to A_2, A_2 is homologous to $A_3, \ldots, A_n$ is homologous to A_1. In this case, then, the number n of components is exactly the divisibility of α.

4. Sufficiency for independent homology classes

In this section we complete the proof of Theorem 2, dealing with the case of a set of homology classes consisting of independent elements.

LEMMA 4.1. *Let F be a closed orientable surface and let $\alpha_1, \ldots, \alpha_n \in H_1(F)$ be independent homology classes that span a summand of $H_1(F)$ on which the intersection pairing of F vanishes. Then there exists $\gamma \in H_1(F)$ such that $\gamma \cdot \alpha_n = 1$ and $\gamma \cdot \alpha_i = 0$ for $i < n$.*

Proof. This is a consequence of Poincaré Duality.

PROPOSITION 4.2. *Let F be a closed orientable surface and let $\alpha_1, \ldots, \alpha_n \in H_1(F)$ be independent homology classes that span a summand of $H_1(F)$ on which the intersection pairing of F vanishes. Then there exist pairwise disjoint simple closed curves $A_1, \ldots, A_n$ in F representing the homology classes $\alpha_1, \ldots, \alpha_n$.*

Proof. The proof will proceed by induction on n. The case $n = 1$ is given by Proposition 3.3.

Now inductively consider the case of $n > 1$ homology classes. By Proposition 3.3 we can find a simple closed curve A_n in F representing α_n. We claim that there is a simple closed curve B_n in F representing a homology class β_n such that B_n meets A_n in exactly one point and such that $[B_n] \cdot \alpha_i = 0$ for $i < n$. By Lemma 4.1 there is a homology class $\gamma_n \in H_1(F)$ such that $\alpha_i \cdot \gamma_n = \delta_{i,n}$. We begin by representing γ_n by a simple closed curve B transverse to A_n. By tubing together neighboring pairs of intersection of B with A_n of opposite sign we can transform B into a disjoint union B'