

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 42 (1996)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE ZERO-IN-THE-SPECTRUM QUESTION
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Kapitel: 5. Universal Covers
DOI: <https://doi.org/10.5169/seals-87882>

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5. UNIVERSAL COVERS

Suppose that M is the universal cover of a compact Riemannian manifold X . We give M the pulled-back Riemannian metric and consider the Laplace-Beltrami operator Δ_p on M . There are numerical invariants derived from $\{\Delta_p\}_{p \geq 0}$, the so-called L^2 -Betti numbers $\{b_p^{(2)}(X)\}_{p \geq 0}$ and Novikov-Shubin invariants $\{\alpha_{p+1}(X)\}_{p \geq 0}$. The L^2 -Betti numbers lie in $[0, \infty)$ and the Novikov-Shubin invariants lie in $[0, \infty] \cup \infty^+$. Here ∞^+ is a formal symbol which should be considered to be greater than ∞ . Roughly speaking, $b_p^{(2)}(X)$ measures the size of $\text{Ker}(\Delta_p)$ as a $\pi_1(X)$ -module and $\alpha_{p+1}(X)$ measures the thickness near zero of the spectrum of Δ_p on $\Lambda^p(M)/\text{Ker}(d)$; the larger $\alpha_{p+1}(X)$, the thinner the spectrum near zero. We refer to [21, 22, 26] for the definitions of these invariants. We will only need the following properties:

PROPERTIES.

1. $b_p^{(2)}(X) = 0 \iff \text{Ker}(\Delta_p) = 0$.
2. $0 \notin \sigma(\Delta_p \text{ on } \Lambda^p(M)/\text{Ker}(d)) \iff \alpha_{p+1} = \infty^+$.
3. $b_p^{(2)}(X)$ and $\alpha_p(X)$ are homotopy-invariants of X .
4. $b_0^{(2)}(X)$, $b_1^{(2)}(X)$, $\alpha_1(X)$ and $\alpha_2(X)$ only depend on $\pi_1(X)$.
5. $b_0^{(2)}(X) = 0$ if and only if $\pi_1(X)$ is infinite.
6. $\alpha_1(X) = \infty^+$ if and only if $\pi_1(X)$ is finite or nonamenable.
7. The Euler characteristic of X satisfies

$$(5.1) \quad \chi(X) = \sum_p (-1)^p b_p^{(2)}(X)$$

8. If X^n is closed then $b_{n-p}^{(2)}(X) = b_p^{(2)}(X)$.
9. If X^{4k} is closed then there are nonnegative numbers $b_{2k,\pm}^{(2)}(X)$ such that $b_{2k}^{(2)}(X) = b_{2k,+}^{(2)}(X) + b_{2k,-}^{(2)}(X)$ and the signature of X satisfies

$$(5.2) \quad \tau(X) = b_{2k,+}^{(2)}(X) - b_{2k,-}^{(2)}(X).$$

One can extend properties 1-7 from compact Riemannian manifolds X to finite CW-complexes K .

In what follows, Γ will denote a finitely-presented group. Given a presentation of Γ , there is an associated 2-dimensional CW-complex K which we call the *presentation complex*. To form it, make a bouquet of circles indexed by the generators of Γ . Attach 2-cells based on the relations of Γ . (We allow trivial or repeated relations in the presentation.) This is the presentation complex.

DEFINITION 7. Put $b_0^{(2)}(\Gamma) = b_0^{(2)}(K)$, $b_1^{(2)}(\Gamma) = b_1^{(2)}(K)$, $\alpha_1(\Gamma) = \alpha_1(K)$ and $\alpha_2(\Gamma) = \alpha_2(K)$.

By Property 4 above, Definition 7 makes sense in that the choice of presentation of Γ does not matter.

As the invariants $b_0^{(2)}(\Gamma)$, $b_1^{(2)}(\Gamma)$, $\alpha_1(\Gamma)$ and $\alpha_2(\Gamma)$ will play an important role, let us state explicitly what they measure. First, from Property 5, $b_0^{(2)}(\Gamma)$ tells us whether or not Γ is infinite. In general, $b_0^{(2)}(\Gamma) = \frac{1}{|\Gamma|}$. Next, from Property 1, $b_1^{(2)}(\Gamma)$ tells us whether or not M has square-integrable harmonic 1-forms (or \tilde{K} has square-integrable harmonic 1-cochains). From Property 2, $\alpha_1(\Gamma)$ tells us whether or not the Laplacian Δ_0 , acting on functions on M , has a gap in its spectrum away from zero. In fact, Property 6 is just a restatement of Corollary 3. Finally, from Property 2, $\alpha_2(\Gamma)$ tells us whether or not the spectrum of the Laplacian on $\Lambda^1(M)/\text{Ker}(d)$ goes down to zero.

5.1 BIG AND SMALL GROUPS

Let us first introduce a convenient terminology for the purposes of the present paper.

DEFINITION 8. The group Γ is big if it is nonamenable, $b_1^{(2)}(\Gamma) = 0$ and $\alpha_2(\Gamma) = \infty^+$. Otherwise, Γ is small.

We recall that Δ_p denotes the Laplace-Beltrami operator on the universal cover M .

PROPOSITION 11. Let X and M be as above. The group $\pi_1(X)$ is small if and only if $0 \in \sigma(\Delta_0)$ or $0 \in \sigma(\Delta_1)$.

Proof. This follows immediately from Properties 1, 2, 4, 5 and 6 above. \square

The question arises as to which groups are big and which are small. Clearly any amenable group is small.

PROPOSITION 12. *Fundamental groups of compact surfaces are small.*

Proof. Suppose that Σ is a compact surface and $\Gamma = \pi_1(\Sigma)$. If Σ has boundary then Γ is a free group F_j on some number j of generators. If $j = 0$ or $j = 1$ then Γ is amenable. If $j > 1$ then $b_1^{(2)}(\Gamma) = j - 1 > 0$.

Suppose now that Σ is closed. If $\chi(\Sigma) \geq 0$ then Γ is amenable. If $\chi(\Sigma) < 0$ then $b_1^{(2)}(\Gamma) = -\chi(\Sigma) > 0$. \square

We now extend Proposition 12 to 3-manifold groups. We use some facts about compact connected 3-manifolds Y , possibly with boundary. (See, for example, [21, Section 6]). Again, all of our manifolds are assumed to be oriented. First, Y has a decomposition as a connected sum $Y = Y_1 \# Y_2 \# \dots \# Y_r$ of *prime* 3-manifolds. A prime 3-manifold is *exceptional* if it is closed and no finite cover of it is homotopy-equivalent to a Seifert, Haken or hyperbolic 3-manifold. No exceptional prime 3-manifolds are known and it is likely that there are none.

PROPOSITION 13 (Lott-Lück). *Suppose that Y is a compact connected oriented 3-manifold, possibly with boundary, none of whose prime factors are exceptional. Then $\pi_1(Y)$ is small.*

Proof. We argue by contradiction. Suppose that $\pi_1(Y)$ is big. First, $\pi_1(Y)$ must be infinite. If ∂Y has any connected components which are 2-spheres then we can cap them off with 3-balls without changing $\pi_1(Y)$. So we can assume that ∂Y does not have any 2-sphere components. In particular, $\chi(Y) = \frac{1}{2}\chi(\partial Y) \leq 0$. From [21, Theorem 0.1.1],

$$(5.3) \quad b_1^{(2)}(Y) = (r - 1) - \sum_{i=1}^r \frac{1}{|\pi_1(Y_i)|} - \chi(Y).$$

As this must vanish, we have $\chi(Y) = 0$ and either

1. $\{|\pi_1(Y_i)|\}_{i=1}^r = \{2, 2, 1, \dots, 1\}$ or
2. $\{|\pi_1(Y_i)|\}_{i=1}^r = \{\infty, 1, \dots, 1\}$.

It follows that ∂Y is empty or a disjoint union of 2-tori. As there are no 2-spheres in ∂Y , if $|\pi_1(Y_i)| = 1$ then Y_i is a homotopy 3-sphere. Thus Y is homotopy-equivalent to either

1. $\mathbf{RP}^3 \# \mathbf{RP}^3$ or
2. A prime 3-manifold Y' with infinite fundamental group whose boundary is empty or a disjoint union of 2-tori.

If Y is homotopy-equivalent to $\mathbf{R}P^3 \# \mathbf{R}P^3$ then $\pi_1(Y)$ is amenable, which is a contradiction. So we must be in the second case. Using Property 3, we may assume that $Y = Y'$. Then as Y is prime, it follows from [24, Chapter 1] that either $Y = S^1 \times D^2$ or Y has incompressible (or empty) boundary. If $Y = S^1 \times D^2$ then $\pi_1(Y)$ is amenable. If Y has incompressible (or empty) boundary then from [21, Theorem 0.1.5], $\alpha_2(Y) \leq 2$ unless Y is a closed 3-manifold with an \mathbf{R}^3 , $\mathbf{R} \times S^2$ or Sol geometric structure. In the latter cases, Γ is amenable. Thus in any case, we get a contradiction. \square

The next proposition gives examples of big groups.

PROPOSITION 14.

1. *A product of two nonamenable groups is big.*
2. *If Y is a closed nonpositively-curved locally symmetric space of dimension greater than three, with no Euclidean factors in \tilde{Y} , then $\pi_1(Y)$ is big.*

Proof. 1. Suppose that $\Gamma = \Gamma_1 \times \Gamma_2$ with Γ_1 and Γ_2 nonamenable. Then Γ is nonamenable. Let K_1 and K_2 be presentation complexes with fundamental groups Γ_1 and Γ_2 , respectively. Put $K = K_1 \times K_2$. Then $\Gamma = \pi_1(K)$. Let $\Delta_p(\tilde{K})$, $\Delta_p(\tilde{K}_1)$ and $\Delta_p(\tilde{K}_2)$ denote the Laplace-Beltrami operator on p -cochains on \tilde{K} , \tilde{K}_1 and \tilde{K}_2 , respectively, as defined in Subsection 5.2 below. Then

$$(5.4) \quad \inf(\sigma(\Delta_1(\tilde{K}))) = \min(\inf(\sigma(\Delta_1(\tilde{K}_1))) + \inf(\sigma(\Delta_0(\tilde{K}_2))), \\ \inf(\sigma(\Delta_0(\tilde{K}_1))) + \inf(\sigma(\Delta_1(\tilde{K}_2)))) > 0.$$

Using Proposition 11, the first part of the proposition follows.

2. If \tilde{Y} is irreducible then part 2. of the proposition follows from the second remark after Proposition 7. If \tilde{Y} is reducible then we can use an argument similar to (5.4). \square

REMARK. Let Γ be an infinite finitely-presented discrete group with Kazhdan's property T. From [6, p. 47], $H^1(\Gamma; \ell^2(\Gamma)) = 0$. This implies that Γ is nonamenable and $b_1^{(2)}(\Gamma) = 0$. We do not know if it is necessarily true that $\alpha_2(\Gamma) = \infty^+$.

5.2 TWO AND THREE DIMENSIONS

In this subsection we relate the zero-in-the-spectrum question to a question in combinatorial group theory. Let K be a finite connected 2-dimensional

CW-complex. Let \tilde{K} be its universal cover. Let $C^*(\tilde{K})$ denote the Hilbert space of square-integrable cellular cochains on \tilde{K} . There is a cochain complex

$$(5.5) \quad 0 \longrightarrow C^0(\tilde{K}) \xrightarrow{d_0} C^1(\tilde{K}) \xrightarrow{d_1} C^2(\tilde{K}) \longrightarrow 0.$$

Define the Laplace-Beltrami operators by $\Delta_0 = d_0^* d_0$, $\Delta_1 = d_0 d_0^* + d_1^* d_1$ and $\Delta_2 = d_1 d_1^*$. These are bounded self-adjoint operators and so we can talk about zero being in the spectrum of \tilde{K} .

PROPOSITION 15. *Zero is not in the spectrum of \tilde{K} if and only if $\pi_1(K)$ is big and $\chi(K) = 0$.*

Proof. Suppose that zero is not in the spectrum of \tilde{K} . From the analog of Proposition 11, Γ must be big. Furthermore, from Properties 1 and 7, $\chi(K) = 0$.

Now suppose that $\pi_1(K)$ is big and $\chi(K) = 0$. From the analog of Proposition 11, $0 \notin \sigma(\Delta_0)$ and $0 \notin \sigma(\Delta_1)$. In particular, $\text{Ker}(\Delta_0) = \text{Ker}(\Delta_1) = 0$. From Properties 1 and 7, $\text{Ker}(\Delta_2) = 0$. As $C^2(\tilde{K}) = \text{Ker}(\Delta_2) \oplus \overline{d_1 C^1(\tilde{K})}$, we conclude that $0 \notin \sigma(\Delta_2)$. \square

Let Γ be a finitely-presented group. Consider a fixed presentation of Γ consisting of g generators and r relations. Let K be the corresponding presentation complex. Then $\chi(K) = 1 - g + r$. Thus zero is not in the spectrum of \tilde{K} if and only if $\pi_1(K)$ is big and $g - r = 1$.

Recall that the *deficiency* $\text{def}(\Gamma)$ is defined to be the maximum, over all finite presentations of Γ , of $g - r$. If $b_1^{(2)}(\Gamma) = 0$ then from the equation

$$(5.6) \quad \chi(K) = 1 - g + r = b_0^{(2)}(\Gamma) - b_1^{(2)}(\Gamma) + b_2^{(2)}(K),$$

we obtain $\text{def}(\Gamma) \leq 1$. This is the case, for example, when Γ is big or when Γ is amenable [5].

As any finite connected 2-dimensional CW-complex is homotopy-equivalent to a presentation complex, it follows from Proposition 15 that the answer to the zero-in-the-spectrum question is “yes” for universal covers of such complexes if and only if the following conjecture is true.

CONJECTURE 1. *If Γ is a big group then $\text{def}(\Gamma) \leq 0$.*

REMARK. If $\pi_1(K)$ has property T then the ordinary first Betti number of K vanishes [6], and so $\chi(K) = 1 + b_2(K) > 0$. Thus zero lies in the spectrum of \tilde{K} .

Now let Y be a 3-manifold satisfying the conditions of Proposition 13. If $\partial Y \neq \emptyset$, we define Δ_p on \tilde{Y} using absolute boundary conditions on $\partial \tilde{Y}$.

PROPOSITION 16. *Zero lies in the spectrum of \tilde{Y} .*

Proof. This is a consequence of Propositions 11 and 13. \square

5.3 FOUR DIMENSIONS

In this subsection we relate the zero-in-the-spectrum question to a question about Euler characteristics of closed 4-dimensional manifolds.

If M is a Riemannian 4-manifold then the Hodge decomposition gives

$$\begin{aligned}
 (5.7) \quad \Lambda^0(M) &= \text{Ker}(\Delta_0) \oplus \Lambda^0(M)/\text{Ker}(d), \\
 \Lambda^1(M) &= \text{Ker}(\Delta_1) \oplus \overline{d\Lambda^0(M)} \oplus \Lambda^1(M)/\text{Ker}(d), \\
 \Lambda^2(M) &= \text{Ker}(\Delta_2) \oplus \overline{d\Lambda^1(M)} \oplus \overline{*d\Lambda^1(M)}, \\
 \Lambda^3(M) &= *\text{Ker}(\Delta_1) \oplus \overline{*d\Lambda^0(M)} \oplus *(\Lambda^1(M)/\text{Ker}(d)), \\
 \Lambda^4(M) &= *\text{Ker}(\Delta_0) \oplus *(\Lambda^0(M)/\text{Ker}(d)).
 \end{aligned}$$

Thus for the zero-in-the-spectrum question, it is enough to consider $\text{Ker}(\Delta_0)$, $\text{Ker}(\Delta_1)$, $\sigma(\Delta_0 \text{ on } \Lambda^0/\text{Ker}(d))$, $\sigma(\Delta_1 \text{ on } \Lambda^1/\text{Ker}(d))$ and $\text{Ker}(\Delta_2)$.

Let Γ be a finitely-presented group. Recall that Γ is the fundamental group of some closed 4-manifold. To see this, take a finite presentation of Γ . Embed the resulting presentation complex in \mathbf{R}^5 and take the boundary of a regular neighborhood to get the manifold.

Now consider the Euler characteristics of all closed 4-manifolds X with fundamental group Γ . Given X , we have $\chi(X \# \mathbf{C}P^2) = \chi(X) + 1$. Thus it is easy to make the Euler characteristic big. However, it is not so easy to make it small. From what has been said,

$$\begin{aligned}
 &\{\chi(X): X \text{ is a closed connected oriented 4-manifold with} \\
 (5.8) \quad &\pi_1(X) = \Gamma\} = \{n \in \mathbf{Z} : n \geq q(\Gamma)\}
 \end{aligned}$$

for some $q(\Gamma)$. *A priori* $q(\Gamma) \in \mathbf{Z} \cup \{-\infty\}$, but in fact $q(\Gamma) \in \mathbf{Z}$ [17, Theorem 1]. (This also follows from (5.9) below.) It is a basic problem in 4-manifold topology to get good estimates of $q(\Gamma)$.

Suppose that $\pi_1(X) = \Gamma$. From Properties 4, 7 and 8 above,

$$(5.9) \quad \chi(X) = 2b_0^{(2)}(\Gamma) - 2b_1^{(2)}(\Gamma) + b_2^{(2)}(X).$$

In particular, if $b_1^{(2)}(\Gamma) = 0$ then $\chi(X) \geq 0$ and so $q(\Gamma) \geq 0$. This is the case, for example, when Γ is big or when Γ is amenable [5].

PROPOSITION 17. *Let X be a closed 4-manifold. Then zero is not in the spectrum of \tilde{X} if and only if $\pi_1(X)$ is big and $\chi(X) = 0$.*

Proof. Suppose that zero is not in the spectrum of \tilde{X} . Then from Proposition 11, $\pi_1(X)$ must be big. Furthermore, $\text{Ker}(\Delta_2) = 0$. From Property 1 and (5.9), $\chi(X) = 0$.

Now suppose that $\pi_1(X)$ is big and $\chi(X) = 0$. From Proposition 11, $0 \notin \sigma(\Delta_0)$ and $0 \notin \sigma(\Delta_1)$. From Property 1 and (5.9), $\text{Ker}(\Delta_2) = 0$. Then from (5.7), zero is not in the spectrum of \tilde{X} . \square

REMARK. If zero is not in the spectrum of \tilde{X} then it follows from Property 9 that in addition, $\tau(X) = 0$. Also, as will be explained later in Corollary 4, if $\pi_1(X)$ satisfies the Strong Novikov Conjecture then $\nu_*([X])$ vanishes in $H_4(B\pi_1(X); \mathbb{C})$.

In summary, we have shown that the answer to the zero-in-the-spectrum question is “yes” for universal covers of closed 4-manifolds if and only if the following conjecture is true.

CONJECTURE 2. *If Γ is a big group then $q(\Gamma) > 0$.*

We now give some partial positive results on the zero-in-the-spectrum question for universal covers of closed 4-manifolds. Recall that there is a notion, due to Thurston, of a manifold having a geometric structure. This is especially important for 3-manifolds. The 4-manifolds with geometric structures have been studied by Wall [32].

PROPOSITION 18. *Let X be a closed 4-manifold. Then zero is in the spectrum of \tilde{X} if*

1. $\pi_1(X)$ has property T or
2. X has a geometric structure (and an arbitrary Riemannian metric) or
3. X has a complex structure (and an arbitrary Riemannian metric).

Proof.

1. If X has property T then the ordinary first Betti number of X vanishes [6]. Thus $\chi(X) = 2 + b_2(X) > 0$. Part 1. of the proposition follows.
2. The geometries of [32] all fall into at least one of the following classes :

- a. $b_0^{(2)} \neq 0 : S^4, S^2 \times S^2, \mathbf{CP}^2$.
- b. $0 \in \sigma(\Delta_0 \text{ on } \Lambda^0 / \text{Ker}(d)) : \mathbf{R}^4, S^3 \times \mathbf{R}, S^2 \times \mathbf{R}^2, \text{Nil}^3 \times \mathbf{R}, \text{Nil}^4, \text{Sol}_0^4, \text{Sol}_1^4, \text{Sol}_{m,n}^4$.
- c. $b_1^{(2)} \neq 0 : S^2 \times H^2$.
- d. $0 \in \sigma(\Delta_1 \text{ on } \Lambda^1 / \text{Ker}(d)) : H^3 \times \mathbf{R}, \widetilde{SL}_2 \times \mathbf{R}, H^2 \times \mathbf{R}^2$.
- e. $\chi > 0 : H^4, H^2 \times H^2, \mathbf{CH}^2$.

Part 2. of the proposition follows.

- 3. Suppose that zero is not in the spectrum of \tilde{X} . From Properties 7 and 9, $\chi(X) = \tau(X) = 0$. From the classification of complex surfaces, X has a geometric structure [32, p. 148–149]. This contradicts part 2. of the proposition. \square

5.4 MORE DIMENSIONS

In this subsection we give some partial positive results about the zero-in-the-spectrum question for covers of compact manifolds of arbitrary dimension. Let us first recall some facts about index theory [18]. Let X be a closed Riemannian manifold. If $\dim(X)$ is even, consider the operator $d + d^*$ on $\Lambda^*(X)$. Give $\Lambda^*(X)$ the \mathbf{Z}_2 -grading coming from (3.12). Then the signature $\tau(X)$ equals the index of $d + d^*$. To say this in a more complicated way, the operator $d + d^*$ defines an element $[d + d^*]$ of the K-homology group $K_0(X)$. Let $\eta : X \rightarrow \text{pt.}$ be the (only) map from X to a point. Then $\eta_*([d + d^*]) \in K_0(\text{pt.})$. There is a map $A : K_0(\text{pt.}) \rightarrow K_0(\mathbf{C})$ which is the identity, as both sides are \mathbf{Z} . So we can say that $\tau(X) = A(\eta_*([d + d^*])) \in K_0(\mathbf{C})$.

We now extend the preceding remarks to the case of a group action. Let M be a normal cover of X with covering group Γ . The fiber bundle $M \rightarrow X$ is classified by a map $\nu : X \rightarrow B\Gamma$, defined up to homotopy. Let \tilde{d} be exterior differentiation on M . Consider the operator $\tilde{d} + \tilde{d}^*$. Taking into account the action of Γ on M , one can define a refined index $\text{ind}(\tilde{d} + \tilde{d}^*) \in K_0(C_r^*\Gamma)$, where $C_r^*\Gamma$ is the reduced group C^* -algebra of Γ .

We recall the statement of the Strong Novikov Conjecture (SNC) [18, 29]. This is a conjecture about a countable discrete group Γ , namely that the assembly map $A : K_*(B\Gamma) \rightarrow K_*(C_r^*\Gamma)$ is rationally injective. Many groups of a geometric origin, such as discrete subgroups of connected Lie groups or Gromov-hyperbolic groups, are known to satisfy SNC. There are no known groups which do not satisfy SNC.

PROPOSITION 19. *Let X be a closed Riemannian manifold with a surjective homomorphism $\pi_1(X) \rightarrow \Gamma$. Let M be the induced normal Γ -cover of X . Suppose that Γ satisfies SNC. Let $L(X) \in H^*(X; \mathbf{C})$ be the Hirzebruch L -class of X and let $*L(X) \in H_*(X; \mathbf{C})$ be its Poincaré dual. Then if $\nu_*(L(X)) \neq 0$ in $H_*(B\Gamma; \mathbf{C})$, zero lies in the spectrum of M . In fact, $0 \in \sigma\left(\Delta_{\frac{\dim(X)}{2}}\right)$ if $\dim(X)$ is even and $0 \in \sigma\left(\Delta_{\frac{\dim(X) \pm 1}{2}}\right)$ if $\dim(X)$ is odd.*

Proof. Suppose first that $\dim(X)$ is even. Suppose that zero does not lie in the spectrum of M . Then the operator $\tilde{d} + \tilde{d}^*$ is invertible. (More precisely, it is invertible as an operator on a Hilbert $C_r^*\Gamma$ -module of differential forms on M .) This implies that $\text{ind}(\tilde{d} + \tilde{d}^*)$ vanishes in $K_0(C_r^*\Gamma)$.

The higher index theorem says that

$$(5.10) \quad \text{ind}(\tilde{d} + \tilde{d}^*) = A(\nu_*([d + d^*])).$$

Let $A_{\mathbf{C}} : K_0(B\Gamma) \otimes \mathbf{C} \rightarrow K_0(C_r^*\Gamma) \otimes \mathbf{C}$ be the complexified assembly map. Using the isomorphism $K_0(B\Gamma) \otimes \mathbf{C} \cong H_{\text{even}}(B\Gamma; \mathbf{C})$, the higher index theorem implies that in $K_0(C_r^*\Gamma) \otimes \mathbf{C}$,

$$(5.11) \quad \text{ind}(\tilde{d} + \tilde{d}^*)_{\mathbf{C}} = A_{\mathbf{C}}(\nu_*(L(X))).$$

By assumption, $A_{\mathbf{C}}$ is injective. This gives a contradiction.

Let T be the operator obtained by restricting $\tilde{d} + \tilde{d}^*$ to

$$\Lambda^{\frac{\dim(X)}{2}}(M) \oplus \overline{\tilde{d}\Lambda^{\frac{\dim(X)}{2}}(M)} \oplus \overline{*d\Lambda^{\frac{\dim(X)}{2}}(M)}.$$

One can show that the other differential forms on M cancel out when computing the rational index of $\tilde{d} + \tilde{d}^*$, so T will have the same index as $\tilde{d} + \tilde{d}^*$. Then the same arguments apply to T to give $0 \in \sigma\left(\Delta_{\frac{\dim(X)}{2}}\right)$.

If $\dim(X)$ is odd, consider the even-dimensional manifold $X' = X \times S^1$ and the group $\Gamma' = \Gamma \times \mathbf{Z}$. As the proposition holds for X' , it must also hold for X . \square

COROLLARY 4. *Let X be a closed Riemannian manifold. Let $[X] \in H_{\dim(X)}(X; \mathbf{C})$ be its fundamental class. Suppose that there is a surjective homomorphism $\pi_1(X) \rightarrow \Gamma$ such that Γ satisfies SNC and the composite map $X \rightarrow B\pi_1(X) \rightarrow B\Gamma$ sends $[X]$ to a nonzero element of $H_{\dim(X)}(B\Gamma; \mathbf{C})$. Let M be the induced normal Γ -cover of X . Then on M , $0 \in \sigma\left(\Delta_{\frac{\dim(X)}{2}}\right)$ if $\dim(X)$ is even and $0 \in \sigma\left(\Delta_{\frac{\dim(X) \pm 1}{2}}\right)$ if $\dim(X)$ is odd.*

Proof. As the Hirzebruch L -class starts out as $L(X) = 1 + \dots$, its Poincaré dual is of the form $*L(X) = \dots + [X]$. The corollary follows from Proposition 19. \square

COROLLARY 5. *Let X be a closed aspherical Riemannian manifold whose fundamental group satisfies SNC. Then on \tilde{X} , $0 \in \sigma\left(\Delta_{\frac{\dim(X)}{2}}\right)$ if $\dim(X)$ is even and $0 \in \sigma\left(\Delta_{\frac{\dim(X) \pm 1}{2}}\right)$ if $\dim(X)$ is odd.*

Proof. This follows from Corollary 4. \square

EXAMPLES.

1. If $X = T^n$ then Corollary 5 is consistent with Example 2 of Section 2.
2. If X is a compact quotient of H^{2n} then Corollary 5 is consistent with Example 3 of Section 2.
3. If X is a compact quotient of H^{2n+1} then Corollary 5 is consistent with Example 4 of Section 2.
4. If X is a closed nonpositively-curved locally symmetric space then Corollary 5 is consistent with the second remark after Proposition 7.

If X is a closed aspherical manifold, it is known that SNC implies that the rational Pontryagin classes of X are homotopy-invariants [18] and that X does not admit a Riemannian metric of positive scalar curvature [29]. Thus we see that these three questions about aspherical manifolds, namely homotopy-invariance of rational Pontryagin classes, (non)existence of positive-scalar-curvature metrics and the zero-in-the-spectrum question, are roughly all on the same footing.

If X is a closed aspherical Riemannian manifold, one can ask for which p one has $0 \in \sigma(\Delta_p)$ on \tilde{X} . The case of locally symmetric spaces is covered by the second remark after Proposition 7. Another interesting class of aspherical manifolds consists of those with amenable fundamental group. By [5], $\text{Ker}(\Delta_p) = 0$ for all p . By Corollary 3, $0 \in \sigma(\Delta_0)$.

PROPOSITION 20. *If X is a closed aspherical manifold such that $\pi_1(X)$ has a nilpotent subgroup of finite index then $0 \in \sigma(\Delta_p)$ on \tilde{X} for all $p \in [0, \dim(X)]$.*

Proof. First, X is homotopy-equivalent to an infranilmanifold, that is, a quotient of the form $\Gamma \backslash G/K$ where K is a finite group, G is the

semidirect product of K and a connected simply-connected nilpotent Lie group and Γ is a discrete cocompact subgroup of G [12, Theorem 6.4]. We may as well assume that $X = \Gamma \backslash G/K$. By passing to a finite cover, we may assume that K is trivial. That is, X is a nilmanifold. From [27, Corollary 7.28], $H^p(X; \mathbb{C}) \cong H^p(g, \mathbb{C})$, the Lie algebra cohomology of g . From [7], $H^p(g, \mathbb{C}) \neq 0$ for all $p \in [0, \dim(X)]$. Thus for all $p \in [0, \dim(X)]$, $H^p(X; \mathbb{C}) \neq 0$.

Now let ω be a nonzero harmonic p -form on X . Let $\pi^*\omega$ be its pullback to \tilde{X} . The idea is to construct low-energy square-integrable p -forms on X by multiplying $\pi^*\omega$ by appropriate functions on X . We define the functions as in [2, §2]. Take a smooth triangulation of X and choose a fundamental domain F for the lifted triangulation of \tilde{X} . If E is a finite subset of $\pi_1(X)$, let χ_H be the characteristic function of $H = \bigcup_{g \in E} g \cdot F$. Given numbers $0 < \epsilon_1 < \epsilon_2 < 1$, choose a nonincreasing function $\psi \in C_0^\infty([0, \infty))$ which is identically one on $[0, \epsilon_1]$ and identically zero on $[\epsilon_2, \infty)$. Define a compactly-supported function f_E on \tilde{X} by $f_E(m) = \psi(d(m, H))$. Then there is a constant $C_1 > 0$, independent of E , such that

$$(5.12) \quad \int_{\tilde{X}} |df_E|^2 \leq C_1 \text{area}(\partial H).$$

Define $\rho_E \in \Lambda^p(\tilde{X})$ by $\rho_E = f_E \cdot \pi^*\omega$. We have $d\rho_E = df_E \wedge \pi^*\omega$ and $d^*\rho_E = -i(df_E)\pi^*\omega$. As f_E is identically one on H , it follows that there is a constant $C > 0$, independent of E , such that

$$(5.13) \quad \frac{\int_{\tilde{X}} [|d\rho_E|^2 + |d^*\rho_E|^2]}{\int_{\tilde{X}} |\rho_E|^2} \leq C \frac{\text{area}(\partial H)}{\text{vol}(H)}.$$

As $\pi_1(X)$ is amenable, by an appropriate choice of E this ratio can be made arbitrarily small. Thus $0 \in \sigma(\Delta_p)$. \square

QUESTION. Does the conclusion of Proposition 20 hold if we only assume that $\pi_1(X)$ is amenable?

6. TOPOLOGICALLY TAME MANIFOLDS

Another class of manifolds for which one can hope to get some nontrivial results about the zero-in-the-spectrum question is given by *topologically tame* manifolds, meaning manifolds M which are diffeomorphic to the interior of a compact manifold N with boundary. If M has finite volume then $\text{Ker}(\Delta_0) \neq 0$,