# IN DEFENSE OF EULER

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# IN DEFENSE OF EULER

by Hans SAMELSON [1])

Can Euler be in need of defense? Well, yes, he is; and it is in the matter of the famous Descartes-Euler theorem, which states that for a (say, convex) bounded polyhedron in 3-space the combination $v - e + f$ of the numbers $v$, $e$, and $f$ of vertices, edges, and faces of the polyhedron (the "Euler characteristic" of the polyhedron) equals 2. (He writes the equation as $S + H = A + 2$.) The "natural" vertices (extreme points), edges, and faces of the polyhedron are understood, although artificial edges — diagonals — occur implicitly.)

He needs defense on two points, one minor and the other major, to be taken up below.

He published two papers on the subject. In the first one, [2], he states the formula, checks it for a number of cases, and says that he has never found a counter example and that he believes it to be true. (He also proves an addition theorem for two polyhedra that have a face in common, and draws many conclusions from the formula.)

In the second one, [3], which appeared right after the first one, he states the formula again and presents a proof for it. We are mainly concerned with the second paper.

First the minor point: It is often overlooked in the literature that he actually stated two theorems. The first one is his formula for the characteristic described above. The second one introduces the "angle sum", say $\omega$, the sum of all the angles of all the faces at their vertices, and states the relation $\omega = 2\pi v - 4\pi$.

In the first paper he derives the second theorem from the first. His argument (after rewriting it in present day terminology) reads as follows: One evaluates $\omega$ by first summing the angles of each face. Let $n$ be the number of sides of

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a given face. The sum of the angles at the vertices of the face is well known to be $(n-2)\pi$. Noting that the sum of the $n$-values for the various faces is $2e$ (since every edge is a side of two faces), one gets for the sum of all these expressions, i.e. for $\omega$, the value $2\pi e - 2\pi f$. This clearly gives his result; in fact, the argument makes the two theorems equivalent, although he doesn't say so.

In the second paper he goes on to say explicitly that the two theorems are equivalent: If either one is true, so is the other one, so that one has to prove only one of them. But then he proceeds to not give the argument for the equivalence, and says that he has decided, on second thought, to prove the two theorems independently. (He doesn't say so, but his two proofs are almost identical — not too surprising in view of the easy equivalence.)

(Unbeknownst to Euler and the whole contemporary mathematical world, the second theorem had been in fact already developed and proved, after a fashion, by Descartes in a note that survives in the form of a handwritten copy by G. W. Leibniz (see, e.g., [8]).)

Is is interesting to compare the two theorems. Although they are so easily transformed into each other, they are quite different in nature. The first one is combinatorial and can be considered as the beginning of algebraic topology. The second one uses the local geometry and can (particularly in Descartes' approach) be thought of as a (polyhedral) version of the Gauss-Bonnet theorem.

The major point of need for defense referred to above has to do with Euler's proof of these theorems. It is generally agreed that his proof is wrong — not only because he failed to notice or state that the theorem applies only to surfaces of genus 0, (piecewise linearly-) homeomorphic to the 2-sphere or the tetrahedron, but mainly because something is wrong with the central step in his argument (see Lebesgue's detailed discussion in [5]; see also [4]).

My intent here is to show that, although this is indeed so, a minor modification, suggested by what we have learned in the mean time, makes the argument into a very good proof for convex polyhedra. The modification consists in making, at a certain point where Euler says to make an arbitrary choice from a given finite set of possibilities, a specific, well defined, and in fact quite obvious, choice which yields again a convex polyhedron. (Lebesgue in [5] develops a very general proof starting from Euler's approach by other considerations.) [After writing this note, I looked once more at the book [7] by J.-C. Pont and found on p. 18, lines 20–23, the statement that Euler's proof would go through, for convex polyhedra, if a convex choice could be shown to exist. Thus my contribution here consists in noting that this choice indeed exists.]

(In passing we note a multilingual curiosity: The person who first stated in print that Euler's formula does not hold for all polyhedra was Lhuilier ([6]) whose name translates into English as Oiler.)

His proof is inductive, on the number $v$ of vertices; the main idea is, given a polyhedron $\mathcal{P}$, to "cut off" a part of $\mathcal{P}$ containing a chosen vertex $O$ in such a way that the remaining polyhedron $\mathcal{P}'$ has exactly all the vertices of $\mathcal{P}$ except $O$ as its vertices and to compare the Euler characteristics and the angle sums $\omega$ of the two polyhedra. In his notation, let $A, B, \dots, F$ be the other vertices on the edges going out from $O$, in the order going around the vertex. Although the circuit $A \dots FA$, which we call $\Gamma$, may not be plane, we can and do introduce enough diagonals like $AC$ to form triangles which will form a surface bounded by the circuit. Over each such triangle there stands a tetrahedron with additional vertex $O$. We now remove these tetrahedra one by one, first $OABC$, then the one whose base has $AC$ as one edge, etc., until all have been cut off. (Euler is quite graphic about this: Put a knife at $B$ and cut all the way to $AC$; then put your knife at $O$ and cut all the way down to $AC$, so that the pyramid $OABC$ comes off.) This leaves us with a polyhedron $\mathcal{P}'$, with one vertex less than the original one, with the triangles $OAB$ etc. replaced by the triangles $ABC$ etc. As Euler notes, some of the triangles like $ABC$ may be coplanar and adjacent to each other and thus combine to form larger faces. Very elementary counting arguments show that the Euler characteristic $v - e + f$ has not changed in the process and that the angle sum $\omega$ has decreased by $2\pi$ and at the same time the number $v$ has also decreased by 1. By iteration we end up with a tetrahedron, for which the Euler formula is true; and so both theorems are proved.

Except that there is trouble: If we allow general, not necessarily convex, polyhedra (of genus 0), then, in the above notation, the plane of a triangle like $ABC$ might contain the point $O$. Or, even if the vertex cone at $O$ is convex, some other part of the polyhedron might pass through the surface formed by the triangles $ABC$ etc.; Euler's procedure then would produce a "polyhedron with self-intersections". One might be able to continue with such polyhedra, but one would certainly need the idea of of immersion of an abstract polyhedron.

A particularly subtle form of this difficulty is what was pointed out by Lebesgue ([4], p. 329) (see also [3]): $\mathcal{P}'$ may not be a polyhedron in the usual sense, but consist of several polyhedra attached to each other along edges or vertices so that the "surface" is no more piecewise linearly equivalent to the 2-sphere (this happens if one of Euler's new diagonals is already an edge of the original polyhedron $\mathcal{P}$); and what one is left with at the end of the iteration may fail to be a tetrahedron. Something like this must in fact happen, if the original polyhedron is not of genus 0. There is even trouble with the counting argument.

Many these difficulties seem to have to do with the lack of convexity. Euler never said anything about his polyhedra being convex or not. (He was familiar with the terms convex and concave, although with a somewhat different sense; at one point in the first paper he divides solids into two classes: those with plane faces and those with convex or concave faces.) It has been suggested that he always had convex polyhedra in mind, without being aware of it. Certainly all his figures in his text are of convex polyhedra, although in his auxiliary constructions he definitely uses non-convex ones, as we saw above.

So let us work with a convex polyhedron (boundary of a polyhedral convex body, the latter being a finite intersection of closed half spaces that happens to be compact with non-empty interior). Then at least there is no trouble at the vertex $O$ of $\mathcal{P}$; Euler's construction applies and there will be no self-intersections. But now the new polyhedron $\mathcal{P}'$ may turn out to be non-convex (the surface formed by the triangles $ABC$ etc., may have "wrinkles" in it); so we are back to Square One.

In fact, Euler was more or less aware of this: He noted that his process of cutting off a corner was not unique, because one can run the diagonals for the circuit $A \dots FA$ in many different ways (unless the vertex $O$ is of order three — exactly three edges going out from it). He shows explicitly, with a figure, the case of an $O$ of order four, where there are two possibilities for a diagonal. He doesn't say so, but one of these choices gives a non-convex polyhedron after his process (unless the four points $A, B, C, D$ are coplanar).

This is how far things are usually taken, implying that Euler's "proof" is hopelessly faulty. But can he really have been that wrong?

It turns out there is a fairly simple way out of this trouble; the idea is to make a (unique) choice of diagonals that results in a convex $\mathcal{P}'$. (As mentioned above, J.-C. Pont in [7] points out that Euler's proof would go through if one could arrange matters so that $\mathcal{P}'$ is convex.) Namely one should define $\mathcal{P}'$ as the boundary of the convex hull of all the vertices of $\mathcal{P}$ except for the chosen vertex $O$. (The fact that Euler didn't think of this choice suggests that he was not thinking of convex polyhedra. But then again the notion of convex hull probably was not around in his time.)

Now it could happen that the vertices of $\mathcal{P}$, except for $O$, all lie in a plane, so that $\mathcal{P}'$ reduces to a plane convex polygon, with $\mathcal{P}$ the cone over it from $O$. There is no difficulty with Euler's formula in that case. (Use the second version of the theorem. Each vertex interior to the base contributes $2\pi$

to the angle sum $\omega$, so one can erase those vertices, as well as the interior edges. And then the result follows from the familiar fact that the sum of the angles of a plane convex $n$-gon is $(n-2)\,\pi$.)

In the "general" case, if $\mathcal{P}'$ is the boundary of a genuine convex body, then clearly the edges $AB$ etc. appear as edges of $\mathcal{P}'$, and the circuit $\Gamma = A \dots FA$ divides the surface of $\mathcal{P}'$ into an old part, say $\Sigma$, "below" and a new part, say $\Sigma'$, "above" the circuit, as seen from $O$. The old part $\Sigma$ is the intersection of $\mathcal{P}$ and $\mathcal{P}'$.

Note that all inner edges of the new part are diagonals in Euler's sense; there are no new vertices. To establish the theorem, we have to compare the new part $\Sigma'$ with the part of $\mathcal{P}$ consisting of the cone from $O$ over $\Gamma$, which we call $\Delta$; we have to show that the contributions of $\Sigma'$ and $\Delta$ to the Euler characteristic are equal; for the other version of the theorem we must show that the angle sum for $\Delta$ is greater by $2\pi$ than that of $\Sigma'$, since clearly $\Delta$ has one more vertex than $\Sigma'$.

For the first part we note that $v - e + f$ for $\Delta$ is clearly 1 (there are as many vertices as edges on $\Gamma$; there are as many edges as faces on the lateral part of $\Delta$; and there is $O$). For $\Sigma'$ we start by cutting it along some diagonal into two parts of the same general nature, and note that the characteristic of $\Sigma'$ equals the sum of the characteristics of the two parts, minus the characteristic of the common diagonal (which appears in both parts); the latter equals 1 (two vertices and one edge). (The "Mayer-Vietoris" relation holds already with "characteristic" replaced by $v$ or $e$ or $f$.) By induction we can assume that the characteristics of the two parts are one (the induction starts with the case of no diagonal, i.e., an ordinary convex polygon); and so that of $\Sigma'$ is also 1.

To make the argument clearer, we could introduce a plane that cuts the cone spanned by $\Delta$ transversally, project $\Sigma'$ onto that plane from $O$, and replace $\Sigma'$ and $\Delta$ by base and lateral side of the pyramid thus formed, with the obvious one-to-one correspondences of vertices, edges, and faces.

For the second version of the theorem we can again use the pyramid just introduced instead of the original figure; angles change under projection, but for any polygonal face the angle sum is the same before and after the projection, since it is determined by the number of edges of the face. In going from the lateral part to the base, the number of vertices goes down by 1, since $O$ disappears. The angle sum of the lateral part is $n\pi$, with $n$ the number of vertices on the circuit $\Gamma$. The angle sum for the base is well known to be $(n-2)\,\pi$, as the sum of the interior angles of a convex polygon of $n$ sides. Thus the angle sum goes down by $2\pi$ on going from $\Delta$ to $\Sigma'$.

Thus the characteristic and the difference $\omega - 2\pi v$ remain constant when one "cuts off a corner". And finally, since $\mathcal{P}'$ is again convex, we can now iterate and end up with a tetrahedron, for which both theorems are clear. So the proofs go through in the convex case, and Euler was on the right track after all.

A last comment on convexity: In the figures in Euler's two papers all the vertex cones appear to be convex. (The vertex cone of a vertex $V$ of a polyhedron $\mathcal{P}$ consists of all rays from $V$ through points of $\mathcal{P}$ "near" $V$.) This property would make the whole polyhedron convex (see Bonnesen-Fenchel [1], p. 3, "Konvexität im Kleinen"). Of course this wasn't known in Euler's time (nor was the question even raised); but he might have felt that it was obvious.

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