VISUALISING MATHEMATICAL CONCEPTS

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Mathematicians may use images in this way to relate ideas in their highly developed cognitive structure. Such *thought experiments* are highly advantageous in contemplating possible relationships before the question of logical proof arises. But it is necessary, as Hadamard said, to be *"guided by images without being enslaved by them"* (ibid, p. 88).

Students do not have such a developed cognitive structure and instead they may be deceived by their imagery. They already have their own concept images developed through previous experience (Tall & Vinner, 1981). Such imagery is often in conflict with the formal theory (see Tall, 1991*a*, 1992 for surveys). Even though concepts are given formal definitions in university mathematics, students may appeal to this imagery and infer theorems through the use of their own thought experiments. For instance, "continuous" might carry the inference of something "going on without a break", so a continuous function must clearly pass through all intermediate values, and must also be bounded and attain its bounds. For a proof by thought experiment, just imagine a picture and see.

VISUALISING MATHEMATICAL CONCEPTS

Although the private images of mathematicians may be difficult to communicate, public images, such as diagrams and graphs enable a great deal of information to be embodied in a single figure. Software which allows visual representations to be controlled by the user, to see dynamic relationships make even more powerful use of visualisation. Having been fascinated by the non-standard idea that a differentiable function infinitely magnified looks like a straight line (within infinitesimals), I wrote computer programs to look at computer drawn graphs under high magnification (figure 1). This allows a visual approach to the notion of differentiability. By using fractals such as the Takagi function (Takagi, 1903) — rechristened the "blancmange" function because of its similarity to a wobbly English milk jelly — functions could be drawn which *never* magnified to look straight (figure 2), hence intimating the notion of a *nowhere differentiable function.* Indeed, a visual proof of this argument is easy to give (Tall, 1982). By taking a small version of the blancmange function $bl(x)$, say $w(x) = bl(1000x)/1000$, for any differentiable function $f(x)$, consider the graph of $f(x) + w(x)$. This looks the same on the computer screen to a normal magnification, but under high magnification (say times 1000), wrinkles appear. This shows visually that, for every differentiable function $f(x)$ there is a non-differentiable function $f(x) + w(x)$, so there are at least as many non-differentiable functions as differentiable ones (figure 3).

A problem with visualisation is that the human mind picks up implicit properties of the imagery and the individual builds up a concept image that incorporates these properties. Graph-plotters tend to draw graphs that consist of continuous parts. So I designed a graph plotter to simulate functions that are different on the rationals and irrationals (Tall 1991*b*, 1993). (The routine uses a continued fraction technique to compute a sequence of rationals approaching a given number and, when a term of the sequence is within ε of the number, it is said to be $(\varepsilon - N)$-pseudo-irrational if the denominator of the fraction exceeds N. By suitably fixing the size of ε and N, computer numbers can be divided into two subsets, (pseudo)-rationals and (pseudo)-irrationals that model various properties of rationals and irrationals.)

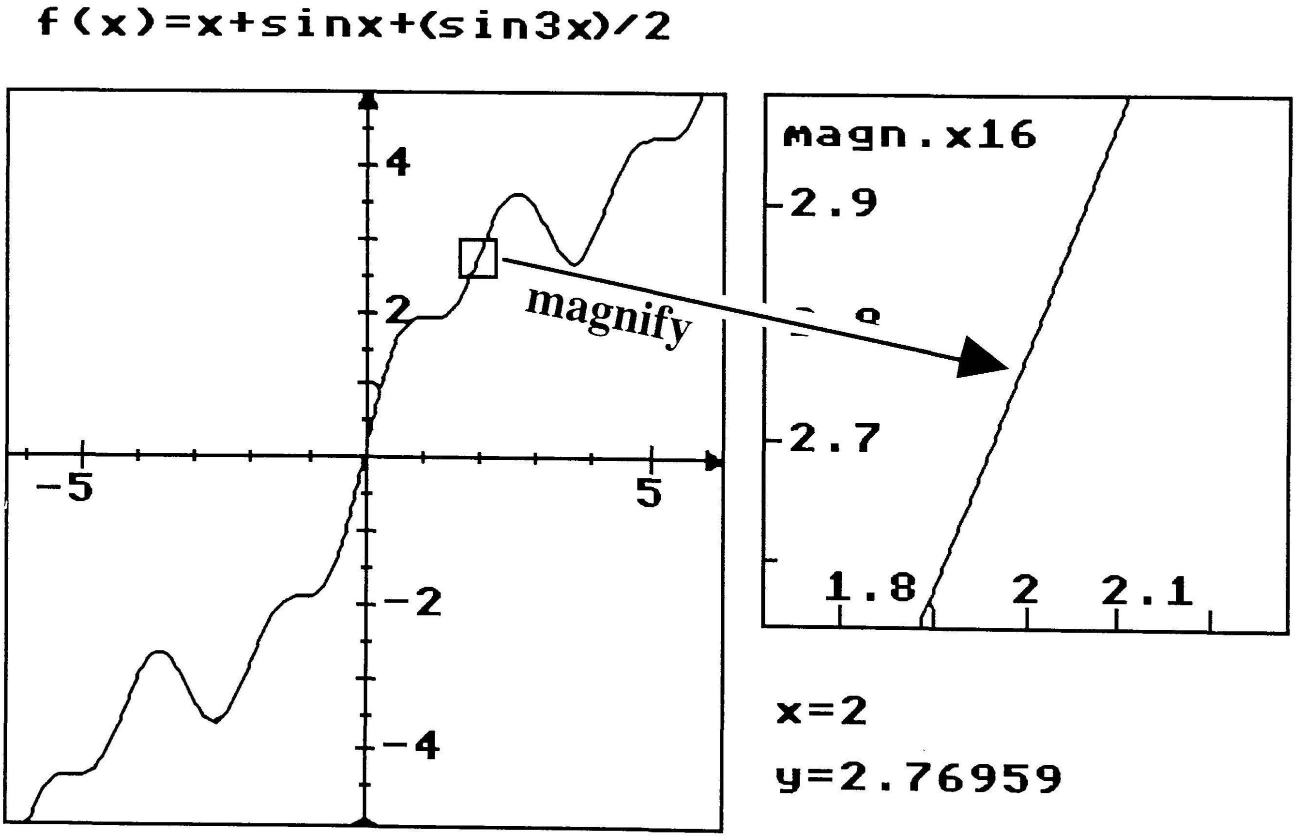

FIGURE 1
Magnifying a locally straight graph

This allows visual insight into more subtle notions. For instance, just as differentiability can be handled visually by magnification maintaining the same relative scales on the axes, continuity can be visualised by maintaining the

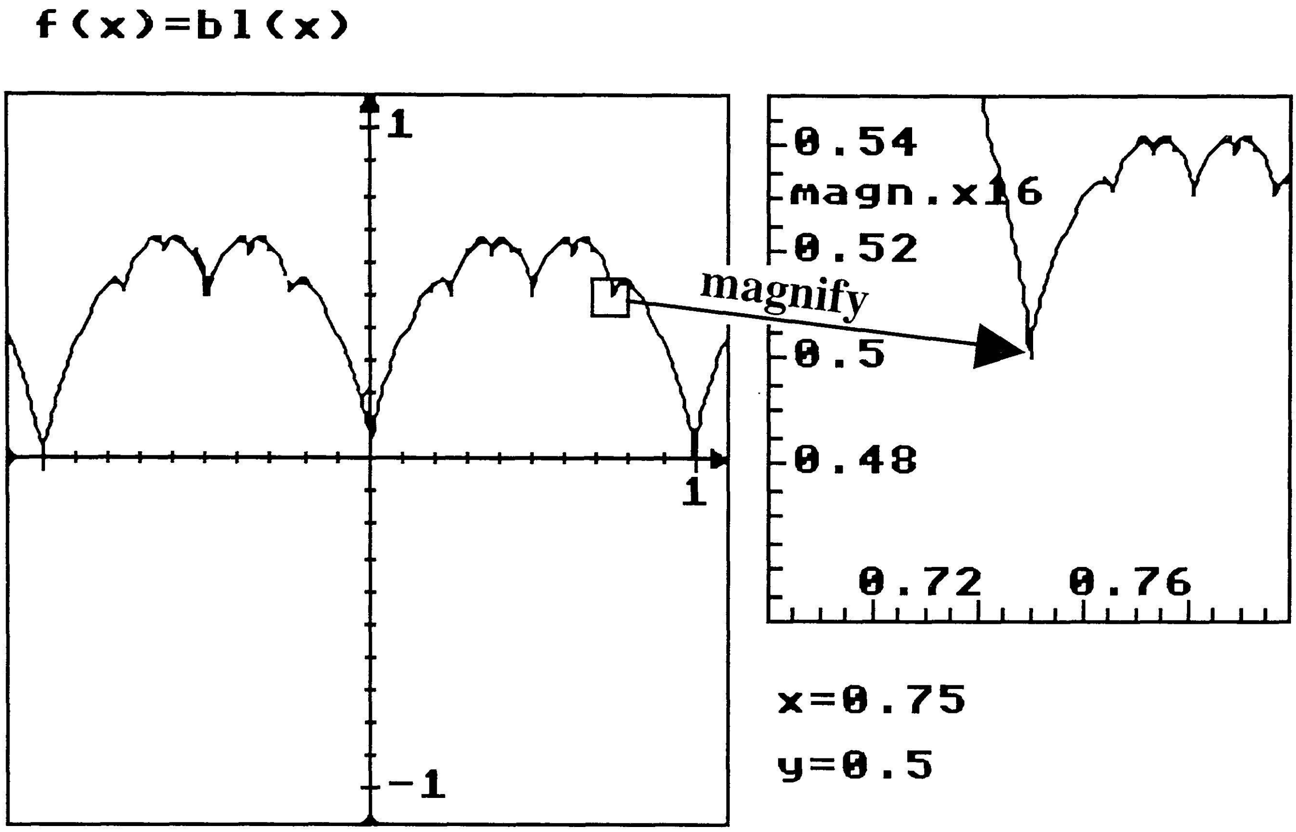

FIGURE 2

Magnifying the nowhere differentiable blancmange

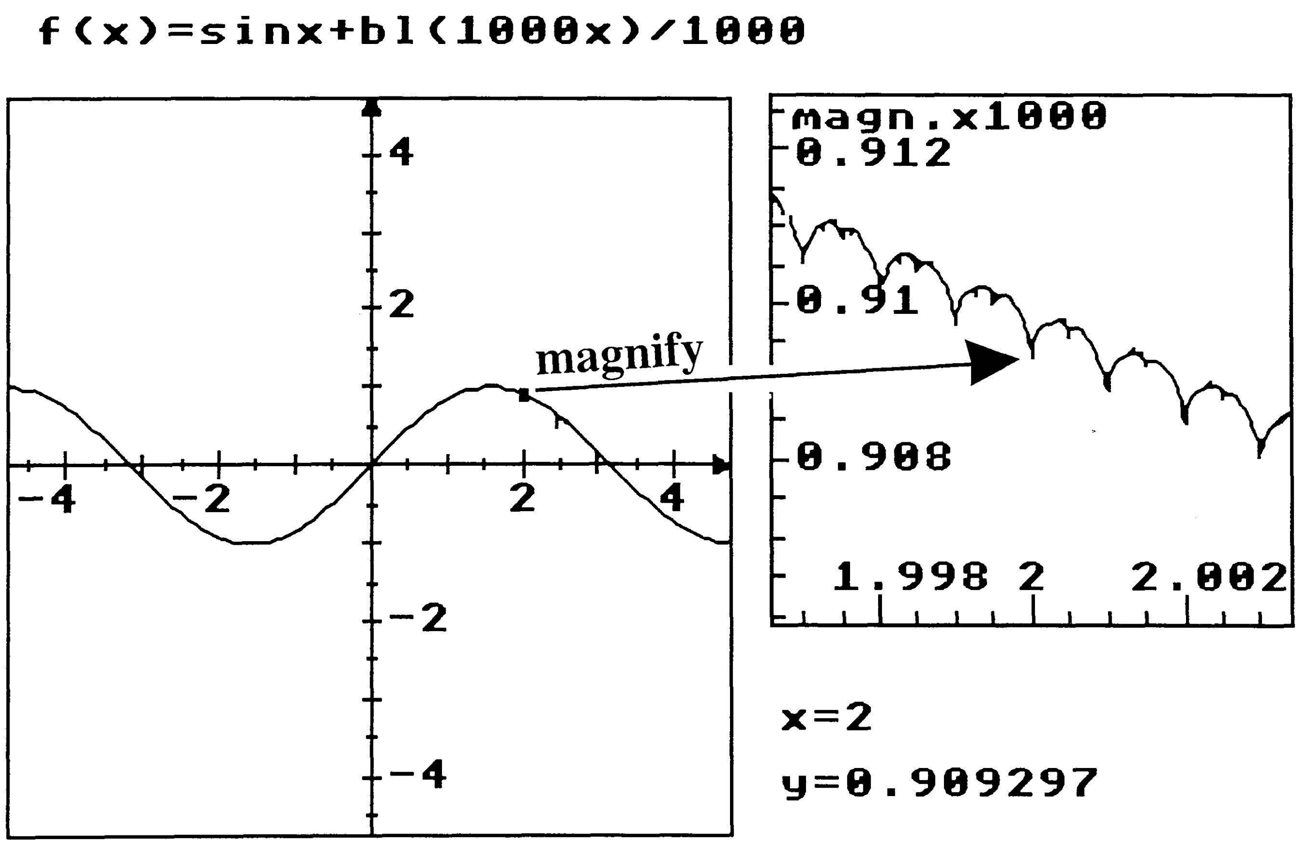

FIGURE 3

Magnifying an interesting graph

vertical scale and stretching the horizontal scale to show less and less of the graph within the same window. A continuous function is one such that any picture of the graph will pull out flat. Figure 4 shows a picture of a graph of a function $f(x)$ which takes the value 1 if x is rational and x^2 if x is irrational. By pulling it horizontal, it is visually continuous at $x = 1$ and $x = -1$, but this clearly fails elsewhere.

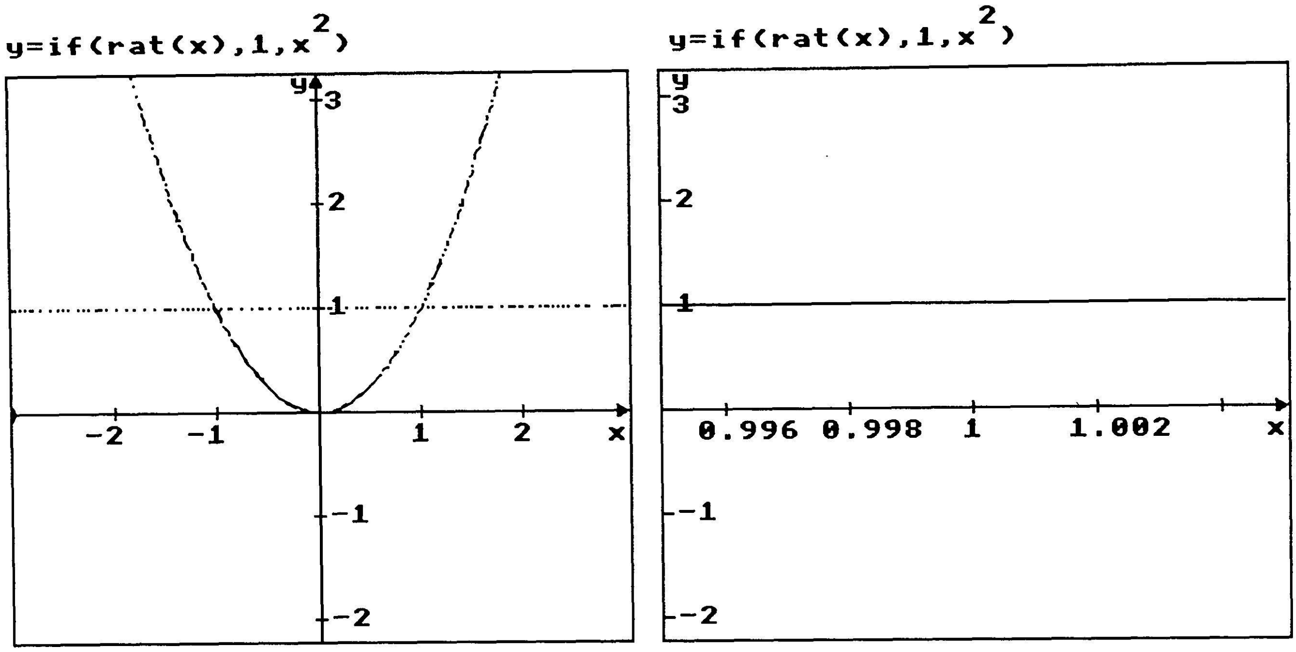

FIGURE 4

Stretching a graph horizontally to "see" if it is continuous

Visual software has been developed in a wide variety of ways, such as Koçak (1986), Hubbard and West (1990) for visualising the solution of differential equations, and a growing mountain of software resources presented each year at the annual *Technology in College Mathematics Teaching* conferences. Such software can give students powerful gestalts to enable them to imagine sophisticated mathematical ideas as simpler visual images. For instance, suppose that a student knows that a differentiable function is locally straight and that a first order differential equation such as $dy/dx = -y$ simply tells the gradient dy/dx of that graph through a point (x, y). Then it is visually clear that a good approximation to the solution can be made by sticking together short straight-line segments with the appropriate gradient. Drawing a picture shows how good this approximation is and visually confirms the existence of a solution, motivating theorems of existence and uniqueness of solutions provided that the gradient is defined along the solution path. This can be valuable both for students who will become mathematics majors and those who will use mathematics in other subjects. I have found such techniques of enormous value teaching science students who have little time for the formal niceties. It proves a good foundation for mathematics majors too, but one must not underestimate the difficulties of linking the visual imagery — which comes as a simultaneous whole — and the logical proofs which involve a different kind of sequential thinking.

USING SYMBOLISM TO COMPRESS PROCESS INTO CONCEPT

Symbols such as $Ax = c$ for a system of linear equations express a relationship in a far more compact form than any corresponding use of natural language. But there is a common use of symbols in mathematics which introduces compression in a subtle way rarely used in ordinary language. It is a method of compression that mathematicians are aware of intuitively but do not articulate in any formal sense, yet it becomes of vital importance in cognitive development. Let me illustrate this with the concept of number and the difference between a mathematician's definition and the cognitive development of the concept.

According to the set-theoretic view of Bourbaki, (cardinal) number concepts are about equivalences between sets. But a set-theoretic approach to number was tried in the "new math" of the sixties and it failed. Why? Almost certainly because the set-theoretic approach is a natural systematisation when everything has been constructed and organised but it is less suitable as the beginning of a *cognitive* development. In essence it is a formulation which is likely to be suggested by experts who have forgotten their earlier development (cognitive principle I) but it proves unsuitable as an approach for the growing individual.

Even though small numbers of two or three objects can be recognised in a glance, cardinal numbers for these and larger numbers begin cognitively in young children as a *process*: the process of counting. Only later do the number symbols become recognised as manipulable number *concepts*.

It often happens that a mathematical process (such as counting) is symbolised, then the symbol is treated as a mathematical concept and itself manipulated as a mental object. Here are just a few examples: